INSTITUTE FOR FUSION STUDIES

DE-FG03-96ER-54346-780 IFSR #780

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March 1997

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Spontaneous Hole-Clump Pair Creation in Weakly Unstable Plasmas

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March 13, 1997

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Abstract

A numerical simulation of a kinetic instability near threshold shows how a hole and clump spontaneously appear in the particle distribution function. The hole and clump support a pair of Bernstein, Greene, Kruskal (BGK) nonlinear waves that last much longer than the inverse linear damping rate while they are upshifting and downshifting in frequency. The frequency shifting allows a balance between the power nonlinearly extracted from the resonant particles and the power dissipated into the background plasma. These waves eventually decay due to phase space gradient smoothing caused by collisionality.

PACS Nos.: 52.35-g, 52.35.Mw, 52.40.Mj

Recently it was observed that a single mode driven unstable by resonant particles can grow explosively to a level that is independent of the closeness to instability threshold set by background dissipation.[1,2] In this problem the wave taps the free energy of a smooth energetically inverted distribution function. Specific examples of this effect include the bump-on-tail instability[3] and the excitation of Alfvén waves in plasmas of interest in fusion research.[4]

An important intrinsic feature of the explosive growth is that the mode frequency shifts (both up and down) from its value at the instability threshold. By the time the amplitude grows to a level where the particle nonlinear trapping frequency ω_b becomes comparable to the linear growth rate (without dissipation) γ_L, the frequency shifts are comparable to γ_L. At this point the explosive behavior described in Ref. 1 is no longer correct, and the mode is expected to saturate. We have developed a simulation code that confirms the expected saturation level, but also reveals a surprise effect: the sideband frequencies continue to shift upward and downward by an amount much larger than γ_L after saturation is reached.

Here we will present the results of the simulation and explain the underlying mechanisms for this effect. We will see that the explosive phase leads to the formation of a phase space hole[5,6]-clump[7–9] pair. In the bump-on-tail instability the hole produces an upshift of the frequency and the clump a downshift of the frequency. We also observe that a hole-clump pair does not emerge far above instability threshold, i.e. when γ_d, the linear damping rate from background dissipation is roughly less than $0.4\gamma_L$.

To develop the nonlinear theory we start from a formalism described in Ref. 2, where the nonlinear response of a single mode is considered and the particle orbits in the absence of perturbations are integrable and periodic, so that an action-angle formalism[10] can be used. The linear mode, with an eigenfrequency ω_0, interacts most strongly with particles that nearly satisfy the resonance condition $\omega_0 = \Omega$, where $\Omega(I)$ is the frequency of the unperturbed motion for a particle with action I, and the resonance condition is satisfied at

$I = I_r$. For nearly resonant particles, one can always choose an action I and conjugate angle ξ in such a way that the perturbed Hamiltonian becomes one-dimensional and reduces to the form,

$$H_1 = 2 \text{ Re} \left[iC(t)V(I_r) \exp\left[i\left(\xi - \omega_0 t\right)\right] \right]. \tag{1}$$

The distribution function, $f(\Omega, \xi, t)$, and the mode amplitude $C(t)$ satisfy the equations,

$$\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \xi} - 2 \text{ Re} \left[iC(t) \left(V \frac{\partial \Omega}{\partial I} \right) \bigg|_{Ir} \exp\left[i(\xi - \omega_0 t) \right] \frac{\partial f}{\partial \Omega} \right] - \nu_{\text{eff}}^3 \frac{\partial^2 f}{\partial \Omega^2} = 0 \tag{2}$$

$$\left(\frac{dC(t)}{dt} + \gamma_d C \right) = \frac{i\omega_0}{G_\omega} \int d\Gamma V^*(I_r) e^{-i(\xi - \omega_0 t)} f \tag{3}$$

where $d\Gamma$ is the six-dimensional phase space volume element and $|C|^2 G_\omega$ is the wave energy per wavelength. The diffusive term represents collisional effects from a Fokker-Planck operator. We solve Eq. (2) subject to the condition that $\frac{\partial f(\Omega)}{\partial \Omega} \to \frac{\partial f_0(\Omega)}{\partial \Omega}$, for Ω-values far from resonance, where f_0 is the equilibrium distribution when $C(t) = 0$.

The linear solution of Eqs. (2) and (3) yields a linear growth rate given by $\gamma = \gamma_L - \gamma_d$ with $\gamma_L = \frac{2\pi^2 \omega_0}{G_\omega} \int d\Gamma_\perp \left(|V|^2 \frac{\partial f_0}{\partial \Omega} \right) \bigg|_{\Omega = \omega_0}$ where $d\Gamma_\perp$ is the 4-dimensional phase space volume element "orthogonal" to (I, ξ).

Here we investigate the behavior of the system when the nonlinear frequency shift of the mode, $\delta\omega = \omega - \omega_0$, can be much larger than γ_L. The formal structure of the theory limits this presentation to cases where $\delta\omega/\omega_0 \ll 1$, so that $\partial f_0/\partial\Omega$ and $V(I)$ do not change significantly, and $\Omega(I) \doteq \Omega(I_r) + (I - I_r)\partial\Omega(I_r)/\partial I$. Observe that without collisions, the trapping frequency, ω_b, of a deeply trapped particle in a constant amplitude wave is given by $\omega_b^2 = 2|C\,V\,\partial\Omega/\partial I|$. It is convenient to define the field amplitude as $A = 2C\,V\,\partial\Omega/\partial I$, at I set equal to a convenient reference choice of the action $I = I^*$. In terms of A, the equations for all physical systems are quite similar.

To simplify the analysis, we now restrict our discussion to a paradigm, the electrostatic bump-on-tail instability. The phase space is then two-dimensional, with $\xi = kx$, $I = mv/k$,

$V = q/k$ where k is a selected quantized wavenumber, v is the particle velocity, and q and m are the particle charge and mass respectively. The quantity ν_{eff}^3 is roughly given by $\nu_{\text{eff}}^3 = \nu_c \omega_0^2$, where ν_c is the pitch angle scattering rate. The frequency ω_0 can be taken equal to the electron plasma frequency which gives $G_\omega \doteq 1$.

Equations (2) and (3) allow $A(t)$ to be real, and this case was taken in the numerical study. Figure 1 shows the solution for $A(t)/\gamma_L^2$ as a function of $\gamma_L t$ for $\gamma_d/\gamma_L = 0.7$ and $\nu_{\text{eff}}^3/\gamma_L^3 = .001$. Initially, the amplitude increases exponentially in time, and then goes into the explosive phase described analytically in Ref. 1 (see below for more details). The explosive solution leads to saturation at a level $A(t)/\gamma_L^2 \approx 1$. It was expected that the instability drive would deplete because of plateau formation in the resonance region, which would then leave only the damping mechanism to absorb the wave energy in a time $\sim 1/\gamma_d$. Instead the envelope of the mode amplitude remains roughly constant in a time interval $1 \ll \gamma_L t \ll (\gamma_L/\nu_{\text{eff}})^3$, and the amplitude oscillates with increasing frequency. Figure 2 shows the upshifted frequency spectrum, $\delta\omega \equiv \omega - \omega_0$, as a function of time; an equal strength downshifted spectrum, $-\delta\omega$, also forms. The most intensive component is the one with the largest frequency shift, but appreciable satellite bands are also generated.

In Fig. 3a we observe the spatially averaged distribution function as a function of time. The depression (enhancement) of the average distribution function coincide with the upshifted (downshifted) frequency $\delta\omega$, which suggests that phase space structures in the form of holes (upshift) and clumps (downshift) have been spontaneously created. This inference is verified in the phase space contour plot shown in Fig. 3b, where the different shades correspond to different values of the distribution f. We see that the values of the distribution at the hole and clump are nearly the same as the value of the creation point. Only later in time, when $\nu_{\text{eff}}^3 t/\gamma_L^2 \approx 1$ does the value of the distribution at the hole and clump begin to change.

The numerical results can be understood as follows: In Ref. 2 it was found that a suffi-

ciently small real amplitude $A(t)$ satisfies the nonlinear equation

$$\frac{\partial A}{\partial t} - \gamma A = -\frac{\gamma_L}{2} \int_0^t d\tau \int_0^{\tau_1} d\tau_1 (t-\tau)^2 A(\tau) A(\tau_1) A(\tau + \tau_1 - t) \exp\left[-\nu_{\text{eff}}^3 T^3(\tau, \tau_1, t)\right] \qquad (4)$$

where $\gamma = \gamma_L - \gamma_d$, and $T^3(\tau, \tau_1, t) = (t-\tau)^2 \left(\frac{2}{3}(t-\tau) + \tau - \tau_1\right)$. In the explosive phase where γ and ν_{eff} can be neglected in Eq. (4), the solution is of the form, $A(t)/\gamma_L^2 = \beta\Big\{\alpha + \exp\left(i\sigma \ln\left(1 - t/t_0\right)\right) + \text{ c.c.}\Big\}\Big/[\gamma_L(t_0 - t)]^{5/2}$, with α, β, σ and t_0 appropriate constants. The domain of validity of this solution is $\frac{(\nu_{\text{eff}} + |\gamma|)^{5/2}}{\gamma_L^{1/2}} \ll |A| \ll \gamma_L^2$. In linearly unstable cases, when $\nu_{\text{eff}} \lesssim \gamma$, the explosive solution always develops. It can also develop when $\nu_{\text{eff}} \gtrsim \gamma$ (including the linearly stable case $\gamma < 0$) if a large enough seed fluctuation arises that satisfies the stated inequality. Observe that at the point of breakdown of validity of Eq. (4), the frequency spectrum has both upshifted and downshifted by an amount $\delta\omega \sim \gamma_L$.

The numerical simulation shows that the initial frequency shift that appears during the explosive phase, continues after the mode saturates at a level $|A| \simeq \gamma_L^2$. The upshifted and downshifted frequencies correspond to the phase space hole and clump respectively. The frequency shift $\delta\omega$ increases slowly to values much larger than γ_L, with $\frac{d}{dt}\delta\omega \ll \gamma_L^2$, which allows the use of bounce averaging methods to describe the evolution of the system.

The solution to the problem can then be viewed as a superposition of two BGK waves, with each one represented as Re $A(t)e^{-i\omega_0 t + i\xi} = -\omega_b^2(t)\cos\psi$; $\psi = \xi - \omega_0 t - \int_0^t dt' \delta\omega(t')$. In the limit $\frac{d\omega_b}{dt} \ll \frac{d\delta\omega}{dt} \ll \omega_b^2 \simeq \gamma_L^2$, the perturbation of the passing particle distribution is negligible compared to that of the trapped particle distribution. Then we can bounce average the trapped particle distribution and use that the distribution is continuous at the separatrix between passing and trapped particles, at the value $f = f_0(\omega_0 + \delta\omega)$. We introduce the following action variable for the trapped particles,

$$J = 4\sqrt{2} \int_0^{\psi_{\max}} \frac{d\psi}{2\pi} [E(J) + \omega_b^2 \cos\psi]^{1/2}, \quad \text{where} \quad E(J) = \frac{(\Omega - \omega_0 - \delta\omega)^2}{2} - \omega_b^2 \cos\psi,$$

with $\psi_{\max} = \cos^{-1}(-E/\omega_b^2)$. The equation for $g \equiv f(J) - f_0(\omega_0 + \delta\omega)$, is then found to satisfy

$$\frac{\partial g}{\partial t} - \nu_{\text{eff}}^3 \frac{\partial}{\partial J} \tau J \frac{\partial g}{\partial J} = -\frac{d\delta\omega(t)}{dt} \frac{\partial f_0(\omega_0)}{\partial \Omega} \tag{5}$$

with $\tau = \dfrac{\partial J}{\partial E}$. Initially $g = 0$, and the boundary conditions are $g = 0$ at $J = \dfrac{8\omega_b}{\pi}$ and $\tau J \dfrac{\partial g}{\partial J} = 0$ at $J = 0$.

To complete the formulation of the reduced problem, we need in addition to Eq. (5), two relations to determine $\delta\omega(t)$ and $\omega_b(t)$. One relation comes from the lowest order form of Eq. (3), where γ_d and $\dfrac{d}{dt}\delta\omega$ are neglected (i.e. when we view the wave as a steady BGK[11] mode). We find

$$\delta\omega\, \omega_b^2 \frac{\partial f_0(\omega_0)}{\partial \Omega} = -\frac{4\gamma_L}{\pi^2} \int_{-\omega_b^2}^{\omega_b^2} dE\, g\Big(J(E)\Big) \int_0^{\psi_{\max}} \frac{d\psi \cos\psi}{[2(E + \omega_b^2 \cos\psi)]^{1/2}}. \tag{6}$$

To next order we take into account γ_d and $d\delta\omega/dt$. The formal procedure is equivalent to the following energy argument.[12] The energy absorbed in a time Δt by the background plasma, $2\gamma_d G_w |C|^2 \Delta t$, is balanced by the energy released by the moving phase space structure, $\omega_0 \Delta I N$, where ΔI is the change of the resonant action due to frequency sweeping, and $N \equiv \int d\Gamma (f - f_0)$ is the number of particles in the phase space structure. Then using $\Delta I = \dfrac{\partial I}{\partial \Omega} \dfrac{d\delta\omega}{dt} \Delta t$, we find

$$\frac{\partial f_0(\omega_0)}{\partial \Omega} \gamma_d \omega_b^4 = -\frac{2\gamma_L}{\pi} \frac{d\delta\omega}{dt} \int_0^{8\omega_b/\pi} dJ\, g(J). \tag{7}$$

Equations (5)–(7) can be written in parameter free form, by defining,

$$\widehat{\omega}_b = \frac{\omega_b}{\gamma_L}, \frac{d\phi}{dt} = \nu_{\text{eff}}^3/\omega_b^2,\ \widehat{J} = J/\omega_b,\ \widehat{\tau} = \omega_b \tau,\ \frac{\partial f_0(\omega_0)}{\partial \Omega} G = -g\gamma_L \left(\frac{\nu_{\text{eff}}^3}{\gamma_d \gamma_L^2}\right)^{1/2},\ \delta\widehat{\omega} = \frac{\delta\omega}{\gamma_L}\left(\frac{\nu_{\text{eff}}^3}{\gamma_d \gamma_L^2}\right)^{1/2}.$$

The result is

$$\frac{\partial G}{\partial \phi} - \frac{\widehat{J}}{\widehat{\omega}_b} \frac{d\widehat{\omega}_b}{d\phi} \frac{\partial G}{\partial \widehat{J}} - \frac{\partial}{\partial \widehat{J}} \left(\widehat{\tau} \widehat{J} \frac{\partial G}{\partial \widehat{J}}\right) = \frac{\delta\widehat{\omega}}{d\phi} \tag{8}$$

$$\delta\widehat{\omega}\widehat{\omega}_b = \frac{4}{\pi^2}\int_0^{8/\pi}\frac{d\widehat{J}G(\widehat{J})}{\widehat{\tau}(\widehat{J})}\int_0^{\psi_{\max}}\frac{d\psi\cos\psi}{[2(\widehat{E}(\widehat{J})+\cos\psi)]^{1/2}}; \tag{9}$$

$$\widehat{\omega}_b^5 = \frac{2}{\pi}\frac{d\delta\widehat{\omega}}{d\phi}\int_0^{8/\pi} d\widehat{J}\,G(\widehat{J}) \tag{10}$$

with $\widehat{J} = \frac{2\sqrt{2}}{\pi}\int_0^{\psi_{\max}} d\psi\left[\widehat{E}(\widehat{J})+\cos\psi\right]^{1/2}$, $0 < \widehat{J} < 8/\pi$, $G(\widehat{J}=8/\pi)=0$, and $\widehat{J}\frac{\partial G(\widehat{J}=0)}{\partial\widehat{J}} = 0$. At early times, $G \doteq \delta\widehat{\omega}$, and then Eqs. (6) and (7) have the solution, $\frac{\omega_b}{\gamma_L} = \frac{16}{3\pi^2}$, $\frac{\delta\omega}{\gamma_L} = \frac{16\sqrt{2}(\gamma_d t)^{1/2}}{3\sqrt{3}\,\pi^2}$, which agrees fairly well with the simulation results (see Fig. 2.).

For this reduced formulation the calculation of $\omega_b(t)$ and the frequency shift $\delta\omega(t)$ remains to be performed by numerically solving Eqs. (8)–(10). From dimensional arguments it is already clear that the maximum change in $\delta\omega$ is $\gamma_L\left(\frac{\gamma_d}{\gamma_L}\right)^{1/2}\left(\frac{\gamma_L}{\nu_{\text{eff}}}\right)^{3/2}$, or ω_0 (whichever is less), and that the pulse with $\omega_b \sim \gamma_L$ lasts for a time $t \sim \gamma_L^2/\nu_{\text{eff}}^3$. We have verified in the simulation results that the decay of the mode is only a function of $\nu_{\text{eff}}^3 t/\gamma_L^2$.

In conclusion we have found a spontaneous frequency sweeping effect in numerical simulations of a single mode in a weakly unstable system. We have also presented the theory of this process. The early evolution of the mode is described by a previously found explosive solution to the point where particle trapping is important. This solution initiates frequency shifts that continue after the mode amplitude saturates. An upshifted phase space hole and downshifted phase space clump emerge at the end of the initial phase. Then as the hole and clump evolve adiabatically, the depth of the hole and the height of the clump increase during a major part of the frequency shifting cycle, allowing the frequency to continue to change. Ultimately collisional diffusion leads to the disintegration of the hole-clump pair on a time scale, $\gamma_L^2/\nu_{\text{eff}}^3$. The wave then damps since a weaker phase space structure reduces the frequency sweeping rate, that in turn reduces the rate of free energy extraction from the particle distribution. The mode amplitude must decrease to allow the power dissipated to

match the free energy extraction rate.

When $\gamma_d/\gamma_L \lesssim 0.4$, spontaneous hole-clump pair formation is not observed. We note that if γ_d is too small, one cannot satisfy the inequality $(\nu_{\text{eff}} + |\gamma|)^{5/2}/\gamma_L^{1/2} \ll |A| \ll \gamma_L^2$, needed to achieve the explosive solution of Eq. (4). Without this explosive phase, the hole and clump apparently do not separate, and the distribution function just flattens in the resonance region, locally depleting the instability drive, so that the wave damps in a time γ_d^{-1} after saturation.

In this paper we considered only a single mode. However, if there are other linear modes in the system, holes and clumps will induce "trapped particle" instabilities[13,14] when $\delta\omega \to \Delta\omega$, where $\Delta\omega$ is the frequency separation of the linear modes. One can easily show that the expected instability rate, γ_{sb}, due to the interaction of the phase space structure with an adjacent mode is as large as $\gamma_{\text{sb}} \simeq \left(\frac{\delta\omega}{\gamma_L}\right)^{1/3} \gamma_L$. As this growth rate appreciably exceeds ω_b, interesting questions arise about the integrity of the phase space structures when other modes can be excited. This topic needs further study.

The chirping mechanism described here may have an important application to energy channelling[15] in a fusion system. Even in subcritical regimes, where phase space structures do not spontaneously arise, they can be excited with a relatively small perturbation if the system is not too far from the instability threshold and if the collisionality is sufficiently small. Thus one might be able to extract energy out of energetic charge fusion products in a controlled way, with only a modest input of external power. This interesting potential application needs further study.

Acknowledgments

We would like to thank Dr. J. Candy for supportive discussions. This work is supported by the U.S. Department of Energy contract #DE-FG03-96ER-54346.

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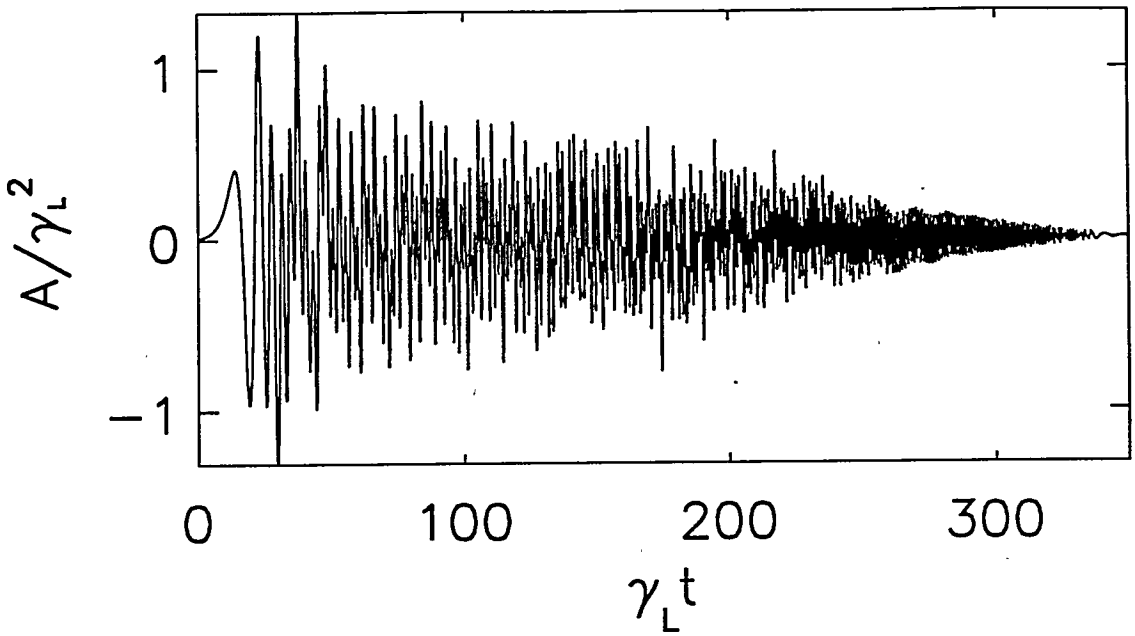
FIGURE CAPTIONS

FIG. 1. Time evolution of normalized mode amplitude for $\gamma_d/\gamma_L = 0.7$, and $(\nu_{\rm eff}/\gamma_L)^3 = 0.001$ (these parameters apply to subsequent figures).

FIG. 2. Contour plots of the evolving Fourier spectrum of $|A(\omega)|^2$ vs. time using a Gaussian time window $\exp\Big(-(t-t_0)^2/\Delta^2\Big)$, with $\Delta = 30\gamma_L^{-1}$. Dotted line is the theory prediction, for early time.

FIG. 3. Particle distribution function with holes and clumps. (a) The spatially averaged distribution as a function of time and $\Omega - \omega_0$. (b) A gray-scale image of the distribution function in phase space at $\gamma_L t = 120$. White corresponds to the smallest values of f and black the largest values. The original resonance is located in the gray area at the mid-line. The islands, corresponding to holes and clumps, are also gray although they are surrounded by other shades of the ambient phase space fluid.

Fig. 1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0
−1
A/γ_L^2
0
100
200
300
$\gamma_L t$

Fig. 2

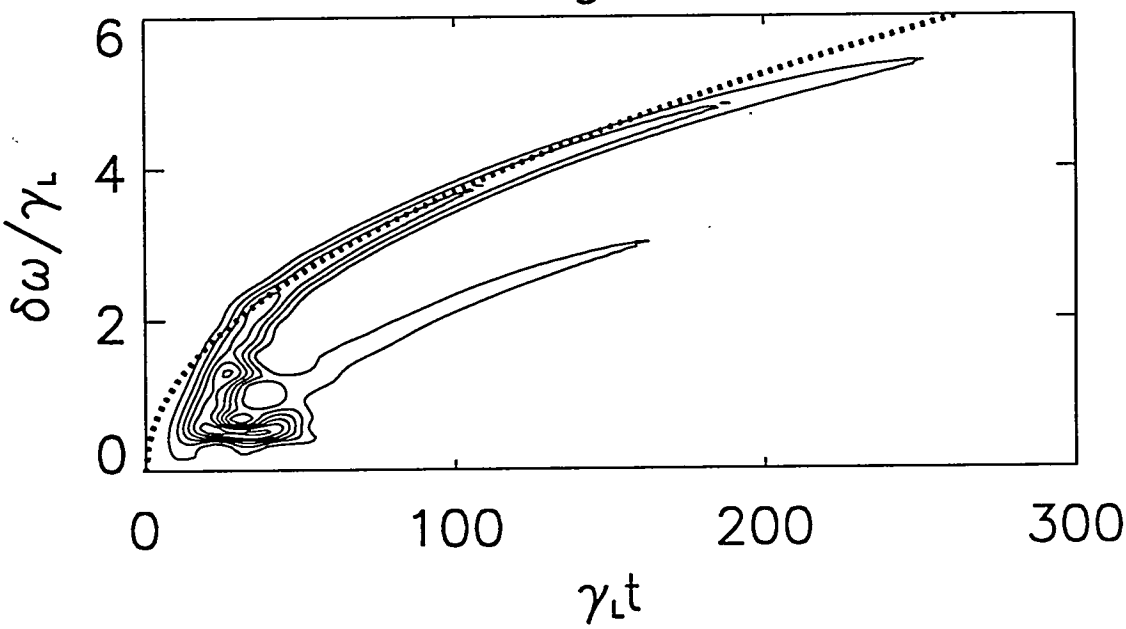
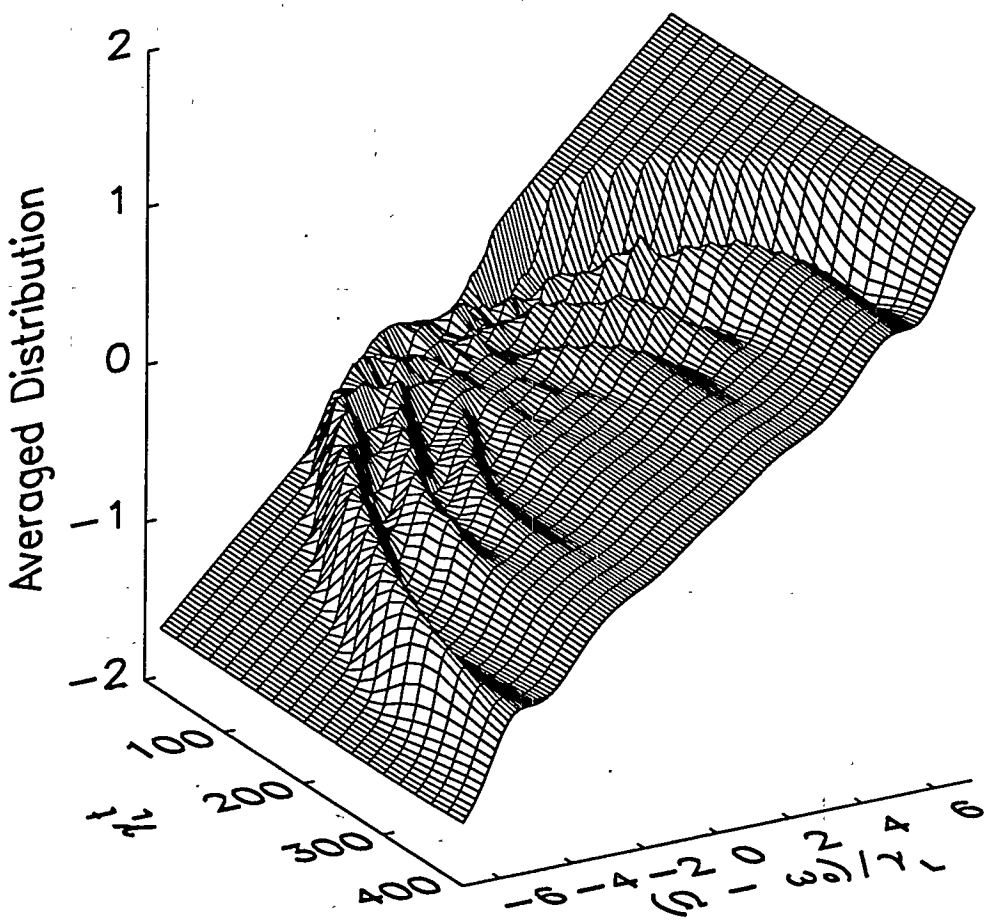

6
4
2
0
$\delta\omega/\gamma_L$
0
100
200
300
$\gamma_L t$

Fig. 3a


Averaged Distribution
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100
200
300
400
$\gamma_2 t$
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$(\Omega - \omega_0)/\gamma_1$
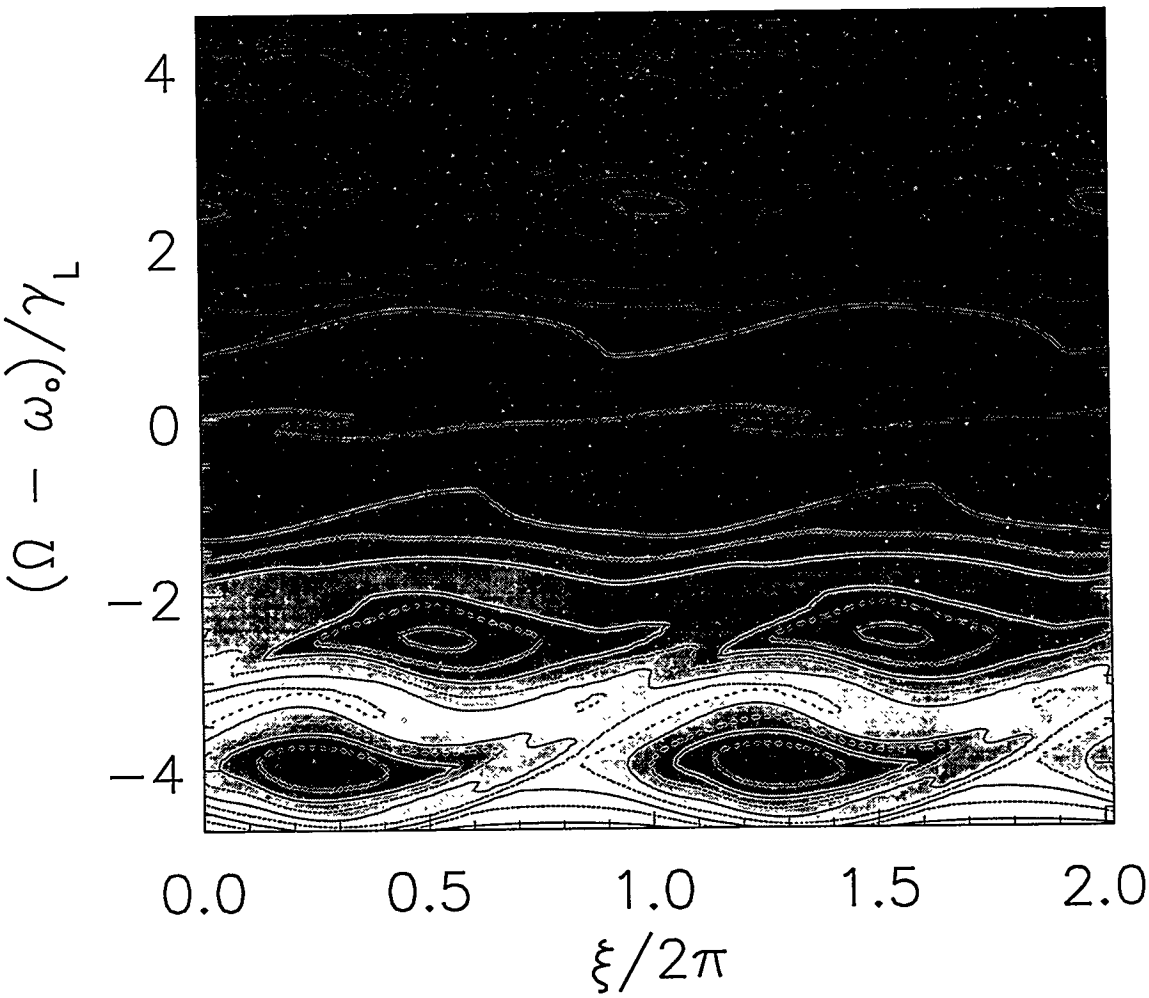
Fig. 3b