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PILE NEUTRON PHYSICS. (Lecture Notes). CHAPTER VIa(cont.). SLOWING DOWN OF NEUTRONS

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CHAPTER VIa (cont.). SLOWING DOWN OF NEUTRONS

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The Space Independent Boltzmann Equation; Energy Distribution of Neutrons

If the production of neutrons throughout all space is uniform, then the neutron density cannot depend on x. The general Boltzmann equation (3.36) reduces to

$$N\sigma F(u,\mu) = -\frac{(1+M)^2}{2M}\sum_{l}\sum_{l'}\frac{2l+1}{2}\,\frac{2l'+1}{2}\int_u^{u+\ln\alpha^2}\int_{-1}^{1} N\sigma_{s_0}(u')F(u',\mu')e^{u'-u}$$

$$f_{l'}(u')\,P_{l'}h(u'-u)\,P_l[g(u'-u)]\,P_l(\mu)\,P_l(\mu')\,du'd\mu' + S(\mu)\,\partial(u-u_0) \quad (3.70)$$

The total neutron flux per logarithmic energy interval, $F_0(u)$, and the total number of neutrons produced per c.c. per second, S_0, are

$$F_0(u) = \int_{-1}^{1} F(u,\mu)d\mu$$

$$S_0 = \int_{-1}^{1} S(\mu)d\mu$$

Thus upon integrating Eq. 3.70 over μ and μ' there results

$$N\sigma F_0(u) = -\frac{(1+M)^2}{2M}\int_u^{u+\ln\alpha^2} N\sigma_{s_0}F_0(u')\,e^{u'-u}\,f[u',h(u'-u)]\,du' + S_0\,\partial(u-u_0) \quad (3.71)$$

In the isotropic scattering case, $f = \frac{1}{2}$, this reduces to

$$N\sigma F_0(u) = -\frac{(1+M)^2}{4M}\int_u^{u+\ln\alpha^2} N\sigma_{s_0}F_0(u')\,e^{u'-u}\,du' + S_0\,\partial(u-u_0)$$

It is to be understood, as before, that $F_0(u) \equiv 0$ if $u < u_0$.

The distribution $F_0(u)$ for large values of u (i.e., at energies far from the source energy) is easy to calculate provided there is no absorption ($N\sigma = N\sigma_{s_0}$).

In this case the distribution equation Eq. 3.71 is

$$N\sigma_{s_0} F_0(u) = -\frac{(1+M)^2}{2M}\int_u^{u+\ln\alpha^2} N\sigma_{s_0} F_0(u')\, e^{u'-u}\, f[u',\, h(u'-u)]\, du' \qquad (3.72)$$

Now, since $f(E',E,\underline{\Omega}',\underline{\Omega})$ is the probability that a collision by a neutron of energy E' and direction $\underline{\Omega}'$ results in an energy E, and a direction $\underline{\Omega}$,

$$\int_{\underline{\Omega}}^{\alpha^2}\int_{E'}^{E'} f(E',D,\underline{\Omega}',\underline{\Omega})\, dE d\underline{\Omega} = 1 \qquad (3.73)$$

hence, on transforming to the variable u, and integrating over angle, we find

$$-\frac{(1+M)^2}{2M}\int_u^{u+\ln\alpha^2} e^{u'-u}\, f[u',h(u'-u)]\, du' = 1 \qquad (3.74)$$

Consequently

$N\sigma_{s_0} F_0(u) = C$, a constant,

satisfies Eq. 3.72. However, since this solution does not satisfy the initial condition (which is derived from Eq. 3.71), namely,

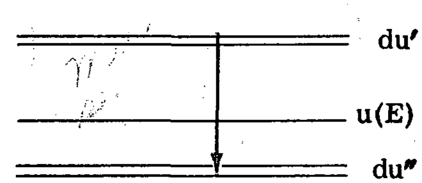
$$\lim_{u \to u_0} N\sigma F_0(u) = S_0\, \partial(u-u_0)$$

it cannot be correct close to the source energy u_0. Thus $N\sigma_{s0} F_0(u) = C$ is only the asymptotic solution of Eq. 3.71, correct at energies u which differ from u_0 by several logarithmic slowing down intervals $\ln\alpha^2$.

We compute the value of the constant C in the case of isotropic scattering, i.e., f = ½. This is of most practical interest since the asymptotic solution applies only to neutrons which have lost considerable energy, and therefore the scattering by the moderator will usually have become isotropic by the time the asymptotic solution becomes valid. To compute C we equate the number of neutrons which cross a given energy E per second per c.c. in the course of slowing down, to the number of neutrons produced per second per c.c.

To calculate the number of neutrons which cross E per second per c.c. we observe that all neutrons from logarithmic interval du′ which enter a logarithmic energy interval du″ lying below u(E) will have crossed E. The number of collisions per logarithmic energy interval du′ per second is

$N\sigma_{s_0} F_0(u')\, du'$;



the probability that these collisions will result in neutrons being thrown into energy interval dE″ is

$$\frac{(1+M)^2}{2M} f[E', \eta(E', E'')]\, dE''$$

or, in the logarithmic energy variable,

$$\frac{(1+M)^2}{2M} e^{u'-u} \quad f[u', h(u'-u'')]\, du''$$

Hence the total number of neutrons thrown across E per second per c.c. is

$$\frac{(1+M)^2}{2M} \int_u^{u+\ln\alpha^2} \int_{u'-\ln\alpha^2}^{u} N\sigma_{s_0} F_0(u')\, e^{u'-u''}\, f[u', h(u'-u'')]\, du''\, du' \tag{3.75}$$

Since the scattering is assumed isotropic, $f = \frac{1}{2}$. Hence the number of neutrons crossing energy E per second per c.c., i.e., the slowing down density q(E), which is to be equated to S_0, is

$$q(E) = S_0 = \frac{(1+M)^2}{4M} C \int_u^{u+\ln\alpha^2} \int_{u'-\ln\alpha^2}^{u} e^{\,u'-u''}\, du''\, du' = C\xi = N\sigma_{s_0} F_0(u)\xi \tag{3.76}$$

Thus

$$C = \frac{S_0}{\xi}$$

and the energy distribution is

$$F_0(u) = \frac{S_0}{N\sigma_{s_0}\xi} \tag{3.77}$$

In terms of E, rather than u, the distribution is

$$F_0(E) = \frac{S_0}{N\sigma_{s_0}\xi E} \tag{3.78}$$

or

$$n(v) = \frac{S_0}{N\sigma_{s_0}\xi v^{3/2}} \tag{3.79}$$

The energy distribution Eq. 3.77 can, of course, be derived from the approximate Eqs. 3.56a and 3.57. In the case of isotropic scattering in a mixture, the slowing down density in an infinite medium in which neutrons are produced uniformly satisfies the equation

$$\frac{\partial}{\partial u}\left[N\sigma_{s_0}\bar{\xi}F_0(u)\right] = S_0\ \partial(u) \tag{3.80}$$

where

$$\bar{\xi} = \sum_i c^i \xi^i$$

Now the solution of Eq. 3.80 is

$$\begin{aligned} N\sigma_{s_0}\bar{\xi}F_0(u) &= S_0 \qquad u > 0 \\ &= 0 \qquad u < 0 \end{aligned} \tag{3.81}$$

and the neutron energy distribution implied by Eq. 3.81 is the same as that of Eq. 3.78.

The assumption of a single substance was not necessary to obtain Eq. 3.81; on the other hand the differential equation which led to Eq. 3.81 is an approximation which was valid, because it involved a Taylor series expansion, only if c^i (in the case of mixtures) varied slowly over one slowing down interval. Actually the energy distribution (Eq. 3.81) is a rigorous asymptotic solution of the space-independent Boltzmann equation only for isotropic scattering in a single substance. For mixtures or anisotropic scattering, Eq. 3.81 is only approximately correct.

Spatial Distribution of Slowed Neutrons; the Slowing Down Kernels

The slowing down density satisfies

$$\Delta q(\underline{r},\tau) = \frac{\partial q(\underline{r},\tau)}{\partial \tau} \tag{3.69}$$

with the initial condition

$$\lim_{\tau\to 0}\ \Delta q(\underline{r},\tau) - \frac{\partial q(r,\tau)}{\partial \tau} = S_0(\underline{r})\ \partial(\tau)$$

Suppose a point source emits one fast neutron per second at $\underline{r} = 0$ in an infinite medium. The slowing down density at some lower energy corresponding to age τ will be the solution of

$$\Delta q(\underline{r},\tau) + \partial(\tau)\ \partial(\underline{r}) = \frac{\partial q(\underline{r},\tau)}{\partial \tau} \tag{3.82}$$

This equation is identical in form with the time dependent diffusion equation discussed in Chapter I. The solution, as was found there, is

$$q(\underline{r},\tau) = \frac{e^{-\frac{r^2}{4\tau}}}{(4\pi\tau)^{3/2}} \tag{3.83}$$

i.e., the slowing down density of neutrons from a point monoenergetic source is distributed around the point according to a Gaussian function. The range r_0 of the Gaussian (i.e., the distance at which the density falls to 1/e of its value at the source) is

$$r_0 = \sqrt{2\tau} \tag{3.84}$$

For many purposes it is important to know the second spatial moment of the slowing down density. If the slowing down density is Gaussian, then the second moment of neutrons slowed to age τ, which we denote by $\overline{r^2}(\tau)$, is

$$\overline{r^2}(\tau) = \frac{\int_0^\infty q(r,\tau) r^2 \cdot 4\pi r^2 dr}{\int_0^\infty q(r,\tau) \cdot 4\pi r^2 dr} = \frac{\int_0^\infty r^4 \, e^{-r^2/4\tau} \, dr}{\int_0^\infty r^2 \, e^{-r^2/4\tau} \, dr} = 6$$

For a Gaussian distribution, the following relation holds between the age, the second moment, and the range:

$$\tau = \overline{r^2}/6 = r_0^2/4$$

The relation between τ and the second moment of the slowing down distribution is the same as the relation between the square of the diffusion length, L, and the second moment of the distribution of thermal neutrons around a point source. For this reason $\sqrt{\tau}$ is often called the slowing down length.

The age τ is related to the logarithmic energy according to Eq. 3.66 by

$$\tau(u) = \int_0^u \frac{du}{3N\sigma_{tr} \, N\sigma_{so}} \tag{3.85}$$

and is therefore a monotone increasing function of u. Thus the spatial distribution of slowed neutrons keeps a Gaussian shape as the neutrons lose energy, but the neutron distribution gradually spreads out since the range r_0 increases with u. The distribution of neutrons slowing down from an energetic source is in this approximation exactly the same as the distribution of heat from an instantaneous heat source.

The energy distribution of the neutrons slowed from a point source is, according to Eq. 3.83

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$$F_0(r,u) = \frac{1}{N\sigma_{s_0} \xi} \frac{e^{-\frac{r^2}{4\tau(u)}}}{[4\pi\tau(u)]^{3/2}}$$

$\tau(u)$ being the function expressed in Eq. 3.85. If all cross sections are constant, then

$$\tau(u) = \frac{u}{3N\sigma_{tr}N\sigma_{s_0}\xi}$$

and

$$F_0(r,u) = \frac{1}{N\sigma_{so}\xi}\,\frac{e^{-\frac{3N\sigma_{tr}N\sigma_{so}\xi r^2}{4u}}}{\left(\frac{4\pi u}{3N\sigma_{tr}N\sigma_{so}\xi}\right)^{3/2}} \tag{3.86}$$

At a given point the flux as a function of u waxes and then wanes; the maximum occurs at the logarithmic energy u_{max} given by

$$u_{max} = \frac{N\sigma_{tr}N\sigma_{so}\xi r^2}{2} \tag{3.87}$$

The slowing down density (Eq. 3.83) from a point monoenergetic neutron source may be designated the "point slowing down kernel." Following the procedure used in the discussion of the time dependent diffusion equation, we can write down the corresponding kernels in other geometries.

From the slowing down density arising from a monochromatic energy source it is a trivial matter to compute the slowing down density from a fission neutron source which is polyenergetic. If the number of neutrons emitted per sec between energy E′ and E′ + dE′ from a point source at the origin is

$$f(E')dE'$$

then the slowing down density from such a source is evidently

$$q[r,\tau(E)] = \int_0^\infty \frac{e^{-\frac{r^2}{4[\tau(E)-\tau(E')]}}}{\{4\pi[\tau(E)-\tau(E')]\}^{3/2}}\,f(E')dE' \tag{3.88}$$

where q $\underline{r}$, $\tau(E)$ is the number of neutrons crossing energy E per second per c.c. at $\underline{r}$. Since in a chain reaction the neutrons originate from a fission spectrum, the slowing down distribution as given by Eq. 3.88 is the one appropriate to a chain reactor in which the moderator is non-hydrogenous.

Slowing Down Kernels

Geometry	Source Normalization	Notation	P
Plane	1 neut/cm^2/sec at (x',τ')	$P_{pl}(x,\tau;\ x',\tau')$	$\dfrac{e^{-\lvert x-x'\rvert^2/4(\tau-\tau')}}{[4\pi(\tau-\tau')]^{1/2}}$
Point	1 neut/sec at (r',τ')	$P_p(\underline{r},\tau;\ \underline{r}',\tau')$	$\dfrac{e^{-\lvert \underline{r}-\underline{r}'\rvert^2/4(\tau-\tau')}}{[4\pi(\tau-\tau')]^{3/2}}$
Line	1 neut/cm/sec at $(r',\ \varphi',\ \tau')$	$P_l(r,\varphi,\tau;\ r',\ \varphi',\tau')$	$\dfrac{e^{-\lvert \rho-\rho'\rvert^2/4(\tau-\tau')}}{4\pi(\tau-\tau')}$, $\rho^2 = r^2 + r'^2 - 2rr' \cos(\varphi-\varphi')$
Spherical Shell	1 neut/sec per shell of radius r' at age τ'.	$P_S(r,\tau;\ r',\tau')$	$\dfrac{1}{4\pi r r'}\left[\dfrac{e^{-\lvert r-r'\rvert^2/4(\tau-\tau')}}{4\pi(\tau-\tau')^{1/2}} - \dfrac{e^{-\lvert r+r'\rvert^2/4(\tau-\tau')}}{4\pi(\tau-\tau')^{1/2}}\right]$
Cylindrical Shell	1 neut/sec/cm of shell of radius r' at age τ'	$P_c(r,\tau;\ r',\tau')$	$\dfrac{e^{-\lvert r-r'\rvert/4(\tau-\tau')}}{4\pi(\tau-\tau')}\ I_0\left(\dfrac{rr'}{2(\tau-\tau')}\right)$

Elementary Improvements on Age Theory

The age approximation, and the Gaussian slowing down distribution which it yields, resulted from a spherical harmonic expansion of the angular distribution, and a Taylor's series expansion of the energy distribution. As has already been pointed out, the Taylor's series expansion is valid only if the mean free path varies slowly over one slowing down interval, while the spherical harmonic expansion could be expected to be good only fairly near the source. Thus the age approximation is poor in hydrogenous media (where the mean free path changes rapidly), or at large distances from the source in any medium.

That the Gaussian cannot be correct at large distances is evident from the following physical argument: Consider neutrons which have made no collisions at all. These will be distributed like $S_0\ (e^{-N\sigma r}/4\pi r^2)$ where $N\sigma$ is the macroscopic scattering cross section and S_0 is the source strength. Now at small distances the Gaussian slowing down distribution will exceed this exponential; at large distances, however, the ratio

$$\frac{\text{source neutrons}}{\text{Gaussian moderated neutrons}} = \left(\frac{e^{-N\sigma r}}{4\pi r^2}\right)\left(\frac{(4\pi\tau)^{3/2}}{e^{-r^2/4\tau}}\right)$$

approaches ∞, since the Gaussian falls off faster than the exponential. Thus at large distances, the distribution is more exponential than Gaussian.

An improvement on the age theory distribution which at least is free from this "first collision paradox" can be expected if the "aging" process, which leads to the Gaussian, is assumed to begin only after the neutrons have made their first collision. The points at which first collisions occur act as "sources" for the slowing down process.

The first collisions are distributed as

$$\frac{e^{-N\sigma(0)r}}{4\pi r^2}$$

the 0 referring to u = 0, the source energy.

According to this picture the slowing down distribution should therefore be

$$q(\underline{r},\tau) = \int_0^\infty \frac{e^{-N\sigma(0)|\underline{r}'|}}{4\pi r'^2} \frac{e^{-\frac{|\underline{r}'-\underline{r}|^2}{4\tau}}}{(4\pi\tau)^{3/2}} d\underline{r}'$$

A further improvement can be made by taking into account the fact that after a neutron has suffered a collision which throws it across energy E, it experiences a "free ride," without changing its energy, until it suffers its next collision. To take this free ride after the last collision into account, it is plausible to include another exponential with mean free path appropriate to the lower energy. Thus the slowing down distribution, including both first and last collisions is

$$q(\underline{r},\tau) = \int_0^\infty \int_0^\infty \frac{e^{-N\sigma(0)|\underline{r}'|}}{4\pi r'^2} \frac{e^{-\frac{|\underline{r}''-\underline{r}'|^2}{4\tau}}}{(4\pi\tau)^{3/2}} \frac{e^{-N\sigma(u)|\underline{r}''-\underline{r}|}}{4\pi|\underline{r}''-\underline{r}|} d\underline{r}'\, d\underline{r}'' \tag{3.89}$$

where $\sigma(0)$ and $\sigma(u)$ are cross sections at the initial and final log energies respectively. Formulas like (3.89) are of course not rigorous; they are rather more plausible than the simple Gaussian and have been used to represent the slowing down distribution from a point monoenergetic source.

In order to compute the second moment of the distribution (3.89) we first state the well known result that the second moment of the distribution from a plane source is just $1/3$ the second moment from a point. This follows from the relation between a point kernel and the corresponding plane kernel,

$$P_p(r) = -\frac{1}{2\pi r} P'_{pl}\ (r).$$

Hence

$$\overline{r^2} = \frac{\int_0^\infty r^2 P_p\ (r) r^2 dr}{\int_0^\infty P_p\ (r) r^2 dr} = \frac{\int_0^\infty Z^3 P'_{pl}\ (z) dz}{\int_0^\infty z P'_{pl}(z) dz}$$

Upon integrating by parts, and using the fact that $r^3 P_{pl}(r) \rightarrow 0$ as $r \rightarrow \infty$ for any kernels of interest, we obtain

$$\overline{r^2} = \overline{3z^2}$$

$\overline{z^2}$ being the second moment of the plane distribution.

With this preliminary we compute $\overline{r^2}$ for the distribution (3.89) by computing the corresponding plane second moment, and multiplying by 3. The plane distribution corresponding to (3.89) we write as

$$q(z,\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_1(z')\ P_{pl}(|z' - z''|)\ K_2(|z'' - z|)\ dz'\ dz'' \tag{3.90}$$

where $K_1(z)$ and $K_2(z)$ are plane transport kernels and P_{pl} is the plane Gaussian kernel. The quantity $q(z,\tau)$ is the convolution of the three kernels K, P_{pl} and K_2.

Now if $\overline{K_1}(B^2)$ is the Fourier transform of $K_1(z)$, i.e., if

$$\overline{K_1}(B^2) = \int_{-\infty}^{\infty} K_1(z)\ e^{iBz} dz$$

then $\overline{z^2_{K_1}}$, the second moment of the distribution defined by $K_1(z)$ is

$$\overline{z^2_{K_1}} = \left.\frac{\dfrac{d^2\overline{K_1}(B^2)}{dB^2}}{\overline{K_1}(B^2)}\right|_{B=0} \tag{3.91}$$

Furthermore, the Fourier transform of $q(z,\tau)$ is

$$\overline{q}(B^2,\tau) = \overline{K_1}(B^2)\ \overline{P}_p\ (B^2)\ \overline{K_2}(B^2) \tag{3.92}$$

as follows from Eq. 3.89 and the definition of the Fourier transform. Hence

$$\left.\frac{\dfrac{d^2\overline{q}(B^2,\tau)}{dB^2}}{\overline{q}(B^2,\tau)}\right|_{B=0} = \left.\frac{\dfrac{d^2\overline{K_1}(B^2)}{dB^2}}{\overline{K_1}(B^2)}\right|_{B=0} + \left.\frac{\dfrac{d^2\overline{P}_{pl}(B^2)}{dB}}{\overline{P}_{pl}(B^2)}\right|_{B=0} + \left.\frac{\dfrac{d^2\overline{K_2}(B^2)}{dB^2}}{\overline{K_2}(B^2)}\right|_{B=0}$$

that is,

$$\overline{z^2_q} = \overline{z^2_{K_1}} + \overline{P^2_{pl}} + z^2_{K_2} \tag{3.93}$$

and

$$\overline{r_q^2} = \overline{r_{K_1}^2} + \overline{r_{P_{pl}}^2} + \overline{r_{K_2}^2}$$

In other words, the second moment of a distribution which is the convolution of several kernels is the sum of the second moments of each kernel.

We now apply this result to the distribution (Eq. 3.90). The second moment of the transport kernel $e^{-N\sigma(0)r}/4\pi r^2$ is

$$\frac{2}{[N\sigma(0)]^2}$$

and the second moment of the Gaussian is 6τ. Hence the second moment of the slowing down distribution corrected for first and last collisions is

$$\overline{r^2_{(\tau)}} = \frac{2}{[N\sigma(0)]^2} + 6\tau + \frac{2}{[N\sigma(u)]^2} \tag{3.94}$$

or, upon using the formula (3.66) for τ,

$$\overline{r^2_{(\tau)}} = \frac{2}{[N\sigma(0)]^2} + 2\int_0^u \frac{du}{N\sigma_{tr}\, N\sigma_{s_0}\xi} + \frac{2}{N\sigma(u)^2} \tag{3.95}$$

The distribution (Eq. 3.90) is unwieldy analytically, and it has therefore been customary to replace it by a single Gaussian

$$\frac{e^{-r^2/4\tau'}}{(4\pi\tau')^{3/2}}$$

where τ', the corrected age, is chosen so as to give the same second moment as (Eq. 3.86). Thus the corrected age is

$$\tau'(u) = \frac{\overline{r^2}}{6} = \frac{1}{3[N\sigma(0)]^2} + \frac{1}{3}\int_0^u \frac{du}{N\sigma_{tr}\, N\sigma_{s_0}\xi} + \frac{1}{3[N\sigma(u)]^2} \tag{3.96}$$

and it is this age, together with the simple Gaussian, which is usually used to represent the slowing down density in a heavy moderator.

The Group Picture

The Gaussian slowing down distribution with the corrected age (Eq. 3.96) is a fairly satisfactory representation of the slowing down process in heavy moderators. However, for certain problems, e.g., those involving slowing down in composite media, even the elegant Gaussian age theory becomes very unwieldy.

The analytical difficulties arise because the age theory equation is a partial differential equation. To avoid these complications a simplified formulation of the slowing down problem which describes the process by a sequence of ordinary differential equations has been used very widely in pile theory.

The general idea of this method, called the method of groups, is to divide the total logarithmic energy interval through which the neutrons pass into a finite number of energy subintervals. Neutrons in a given energy group are supposed to diffuse without energy loss until they have experienced a number of collisions equal to the average number of collisions actually required to pass through the energy interval; at this time they pass into the next lower energy interval. Thus removal from an energy interval is treated as an "adsorption" process, the "absorption" cross-section, σ_a^*, being determined from the relation

$$\frac{\sigma_s}{\sigma_a^*} = \text{number of collisions before removal from energy range.}$$

The cross-section for removal of neutrons from one group is also the cross-section for creation of neutrons in the next lower group. The slowing down density, i.e., the number of neutrons passing from the νth energy group to the ν + 1st group, is therefore

$$q_\nu = N\sigma_a^* \Phi_\nu \tag{3.97}$$

where Φ_ν is the flux of neutrons in the ν^{th} energy group. If the magnitude of a logarithmic energy interval is denoted by u_ν, and ξ is the logarithmic energy decrement per collision, then the number of collisions required to cross u_ν is u_ν/ξ, and the "absorption" cross-section for the ν^{th} group is

$$\sigma_a^* = \frac{\sigma_{s\nu}\,\xi}{u_\nu} \tag{3.98}$$

where $\sigma_{a\nu}$ is the average scattering cross-section in the u_ν energy interval.

If D_ν is the average diffusion coefficient in the ν^{th} energy group, then the neutron flux $\Phi_\nu(\underline{r})$ in the ν^{th} group satisfies the diffusion equation

$$D_\nu \Delta\Phi_\nu(\underline{r}) - N\sigma^*_{a\nu}\, \Phi(\underline{r}) + N\sigma_{a\nu-1}\, \Phi_{\nu-1}(\underline{r}) + S_\nu(\underline{r}) = 0 \tag{3.99}$$

$S_\nu(\underline{r})$ being the number of neutrons produced by an external source per unit volume at $\underline{r}$ in the νth energy interval.

In a one group picture, $\nu = 1$, in which the external source is a ∂-function at the origin, the group equation is

$$D_1\Delta\Phi_1 - N\sigma^*_{a_1}\Phi_1 + \partial(r) = 0 \tag{3.100}$$

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this has the solution

$$N\sigma^*_{a_1} \ \Phi_1(r) = \frac{e^{-r/L_1^*}}{4\pi r L_1^{*2}} \tag{3.101}$$

where

$$L_1^{*2} = D_1/N_1\sigma^*_{a_1} \tag{3.102}$$

The second moment of this distribution is $6L^{*2}$, and the slowing down length is given by

$$\frac{\overline{r^2}}{6} = L_1^{*2} = \frac{D_1^*}{N\sigma^*_{a_1}} \tag{3.103}$$

If we substitute for $N\sigma^*_{a_1}$ and D_1 their expressions in terms of cross-sections, we obtain

$$L_1^{*2} = \frac{u_1}{3N\sigma_{tr_1} N\sigma_{s_1}} \tag{3.104}$$

This is identical with the age theory expression for the second moment provided the product $N\sigma_{tr_1} N\sigma_{s_1}$ is chosen as

$$\frac{1}{N\sigma_{tr_1} N\sigma_{s_1}} = \frac{1}{u_1} \int_0^{u_1} \frac{du}{N\sigma_{tr_1}(u) N\sigma_{s_1}(u)} \tag{3.105}$$

With this choice of average cross sections, the one-group picture is seen to give a spatial distribution which is exponential instead of Gaussian which has the same second moment as the age theory, but

In the n-group picture the distribution from a localized source of high energy neutrons is readily found by solving Eq. 3.99. For simplicity we deal with the problem plane symmetry; the solution from a point source is then found by differentiating according to Eq. 1.141. The differential equations to be solved are

$$\left.\begin{aligned} & D_1 \frac{d^2\Phi_1}{dx^2} - N\sigma^*_{a_1} \ \Phi_1 + \partial(x) = 0 \\ & D_\nu \frac{d^2\Phi_\nu}{dx^2} - N\sigma^*_{a\nu}\Phi_\nu = N\sigma^*_{a\nu-1} \ \Phi_{i-1} = 0 \qquad \nu = 2 \ . \ . \ . \ n. \end{aligned}\right\} \tag{3.106}$$

To solve these equations we make a Fourier transformation,

$$\Phi_\nu(x) = \int_{-\infty}^{\infty} \overline{\Phi}_\nu(\omega) \ e^{i\omega x} d\omega, \quad \partial(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} d\omega$$

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The transforms satisfy

$$(D_1\omega^2 + N\sigma^*_{a_1})\,\overline{\Phi}_1 = 1/2\pi$$
$$-(D_\nu\,\omega^2 + N\sigma^*_{a\nu})\,\overline{\Phi}_\nu + N\sigma^*_{a\nu-1}\overline{\Phi}_{\nu-1} = 0 \qquad \nu > 1$$

Hence

$$\overline{\Phi}_1(\omega) = \frac{1}{2\pi(D_1\omega^2 + N\sigma^*_{a_1})};\ \overline{\Phi}_k(\omega) = \overline{\Phi}_1(\omega)\ldots\overline{\Phi}_{k-1}(\omega) = \frac{1}{2\pi(D_1\omega^2 + N\sigma^*_{a_1})}\prod_{\nu=2}^{k} \times \frac{N\sigma^*_{a\nu-1}}{(D_\nu\omega^2 + N\sigma^*_{a\nu})}$$

and therefore

$$\Phi_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega x}}{D_1\omega^2 + N\sigma^*_{a_1}}\,d\omega$$
$$\Phi_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega x}}{D_1\omega^2 + N\sigma^*_{a_1}}\prod_{\nu=2}^{k}\frac{N\sigma^*_{a\nu-1}}{(D_\nu\,\omega^2 + N\sigma^*_{a\nu})}\,d\omega \qquad (3.107)$$

Since $\Phi_k(\omega)$ is the product of the Fourier transforms of the preceding Φ_ν, the distribution $\Phi_k(x)$ in a given group must be the convolution of the distribution of the previous groups. Physically this means that each group acts as a source for the succeeding group.

To actually compute $\Phi_\nu(x)$, it is necessary to evaluate the integrals in Eq. 3.107. The integrands have poles in the upper half plane at

$$\omega = i\sqrt{\frac{N\sigma_{a\nu}}{D_\nu}} = i/L^*_\nu \qquad (3.108)$$

and we assume for simplicity that all roots are simple. Hence, according to the residue theorem,

$$\Phi_k(x) = \frac{1}{N\sigma^*_{a_k}}\sum_{j=1}^{k}\frac{1}{2L^*_j}\,\frac{e^{-x/L^*_j}}{(1-L^{*2}_1/L^{*2}_j)(1-L^{*2}_2/L^{*2}_j)\ldots\ldots(1-L^{*2}_k/L^{*2}_j)} \qquad (3.109)$$

where the term $(1-L^{*2}_j/L^{*2}_j)$ is omitted from the sum. If $k = 1$ (one group picture), Eq. 3.109 reduces to

$$N\sigma^*_{a_1}\,\Phi_1(x) = \frac{e^{-x/L^*_j}}{2L^*_j} \qquad (3.110)$$

which is the plane equivalent of Eq. 3.101.

Since $\Phi_k(x)$ is the convolution of all the previous $\Phi_\nu(x)$, the total slowing down length for the neutrons slowed out of the k^{th} group must be the sum of the squares of the slowing down lengths in each group individually:

$$\frac{\overline{r_k^2}}{6} = L_1^{*2} + L_2^{*2} + \ldots + L_k^{*2} \tag{3.111}$$

If the number of groups becomes infinite, but each L_ν is reduced so that

$$\frac{\overline{r_\infty^2}}{6} = \sum_{\nu=1}^{\infty} L_\nu^{*2} \equiv \tau \tag{3.112}$$

where τ remains finite, then the group picture should go over into the continuous age theory. The slowing down density in the ∞-group case can be computed most readily by first passing to the limit in the integrand of (3.107) and then evaluating the integral. From Eq. 3.107

$$N\sigma_{a_k}^* \; \Phi_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega x} \, d\omega}{(1 + L_1^{*2}\omega^2)(1 + L_2^{*2}\omega^2)\ldots(1 + L_k^{*2}\omega^2)}$$

Now

$$\lim_{\substack{L_\nu^* \to 0 \\ \Sigma L_\nu^{*2} \to \tau}} \prod_{\nu=1}^{\infty} \frac{1}{(1 + L_\nu^{*2}\omega^2)} = e^{-\omega^2 \Sigma L_\nu^{*2}} = e^{-\omega^2\tau} \tag{3.113}$$

as is verified by taking logarithms of both sides. Hence

$$\lim_{k \to \infty} N\sigma_{a_k}^* \; \Phi_k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \, e^{-\omega^2\tau} = \frac{e^{-x^2/4\tau}}{(4\pi\tau)^{1/2}} \tag{3.114}$$

that is, the group picture and the age picture merge when the number of groups becomes infinite.

The great merit of the group method is that it involves ordinary instead of partial differential equations. By taking enough groups it is possible to approximate the age theory slowing down function to any degree of accuracy, and still deal only with ordinary equations. The approximate slowing down functions which are constructed out of group picture exponentials are called "synthetic" kernels. In pile problems involving H_2O as moderator, it is customary to use one or two fast neutron groups in addition to the thermal neutron group; in piles moderated by heavier materials as many as five or six groups have been used.

In assessing the relative accuracy of the group method and the age theory, it must be remembered that the slowing down function from a point fission source, even in, say, graphite, is not a Gaussian because of the energy spread of the source neutrons. Thus in graphite the three group model is only slightly less accurate than the single Gaussian while in H_2O, because of the very long mean free path at high energies, the slowing down is more nearly represented by a single group picture than by a Gaussian.

Average Transport Cross Section in Group Method

In order to obtain a one group distribution which has the same second moment as the Gaussian, it is necessary to average the product of the transport and scattering cross sections according to Eq. 3.105. In problems involving only one medium it is only this product which determines the neutron distribution. However, in problems involving composite media, since one of the boundary conditions across an interface is continuity of the net current, and the current is proportional to the transport mean free path, it is necessary to find an appropriate average for the transport mean free path separately.

To calculate an average transport mean free path which will ensure continuity of the net neutron current in a group, it is necessary to make some assumption with regard to the actual energy distribution of the neutrons in a given group. Evidently the energy distribution will depend on the particular arrangement and properties of the slowing down media on each side of the boundary. However, as a simple approximation, it is useful to assume that the energy distribution of the neutrons is the asymptotic distribution

$$\Phi(x,E)\,dE = f(x)\,\frac{dE}{N\sigma_{s_0}\xi E} \tag{3.115}$$

where σ_{s_0} is the scattering cross section.

The total flux of neutrons in a group from energy E_1 to E_2 is

$$\Phi(x) = \int_{E_1}^{E_2} \Phi(x,E)dE = f(x)\int_{E_1}^{E_2} \frac{dE}{N\sigma_{s_0}\xi E} \tag{3.116}$$

and the net current is

$$\frac{1}{3}\overline{\lambda}_{tr}\,\frac{d\Phi(x)}{dx} = \frac{1}{3}\int_{E_1}^{E_2} \lambda_{tr}(E)\,\frac{d\Phi}{dx}(x,E)\,dE = \frac{1}{3}\frac{df}{dx}\int_{E_1}^{E_2}\frac{\lambda_{tr}(E)dE}{N\sigma_{s_0}\xi E} \tag{3.117}$$

where $\overline{\lambda}_{tr}$ is the correct average transport mean free path. Thus combining Eqs. 3.116 and 3.117, we obtain

$$\overline{\lambda}_{tr} = \frac{\displaystyle\int_{E_1}^{E_2} \lambda_{tr}(E)\,\frac{dE}{N\sigma_{s_0}E}}{\displaystyle\int_{E_1}^{E_2}\frac{dE}{N\sigma_{s_0}E}} \tag{3.118}$$

if ξ is constant, i.e., the average transport mean free path which will give continuity of flow and density in a group in which the asymptotic energy distribution holds is an average over $1/N\sigma_s E$.

The Energy Transfer Distribution of Slowed Neutrons

It is a matter of some practical importance to calculate the manner in which the energy transferred to a moderator by elastic collisions of fast neutrons is distributed in space as the neutrons slow down from a plane source. If the flux of neutrons of log energy $u = \ln E_0/E$ is $F(x,u)$ (plane symmetry), then the number of elastic collisions per c.c. per second at energy $E = e^u$ is

$$N\sigma_{s_0} F(x,u)$$

Since the logarithm of the ratio of the average energies E′ and E after two successive collisions is

$$\ln E'/E = \xi$$

the average energy loss per collision, ΔE, is

$$\Delta E = E' - E = E(e^{\xi} - 1) \tag{3.119}$$

i.e., if the moderator is heavy,

$$\Delta E \approx \xi E \tag{3.120}$$

This energy increment appears as kinetic energy of the moderator atom. Hence $E(x)$, the energy released per c.c. per second to the moderator by elastic collisions, is, for heavy moderators,

$$E(x) = \int_0^\infty N\sigma_{s_0}(u)\,\xi E\,F(x,u)\,du = E_0 \int_0^\infty N\sigma_{s_0}(u)\,\xi e^{-u}\,F(x,u)\,du \tag{3.121}$$

To evaluate this integral an assumption must be made with respect to the neutron distribution $F(x,u)$. This we take to be Gaussian:

$$F(x,u) = \frac{q(x,u)}{N\sigma_{s_0}\xi} \quad \frac{S}{N\sigma_{s_0}\xi}\,\frac{e^{-K^2/4\tau(u)}}{[4\pi\tau(u)]^{1/2}} \tag{3.122}$$

where S_0 is the number of neutrons emitted per sq. cm. per second by the source. Hence we obtain for the integral (3.121),

$$E(x) = SE_0 \int_0^\infty \frac{e^{-[u+x^2/4\tau(u)]}}{[4\pi\tau(u)]^{1/2}}\,du \tag{3.123}$$

[redacted]

This integral can in general be evaluated only by numerical methods. However, if all cross sections are constant, then, in simplest approximation,

$$\tau(u) = \frac{\lambda_{tr}\lambda}{3} u \qquad (3.124)$$

and

$$E(x) = S_0 E_0 \int_0^{\infty} \frac{e^{-\left[u - \frac{3x^2}{\lambda_{tr}\lambda_u}\right]}}{\left[\frac{4}{3}\pi\lambda_{tr}\lambda_u\right]^{1/2}} du \qquad (3.125)$$

Evaluating the integral according to Watson's Bessel Functions, p. 183, we obtain

$$E(x) = \frac{\alpha S_0 E_0}{2} e^{-\alpha x} \qquad (3.126)$$

where

$$\alpha = [3\xi/\lambda\lambda_{tr}]^{1/2} \qquad (3.127)$$

The total energy emitted from one side of the source plane cm^2 per second is $S_0E_0/2$; thus, according to Eq. 3.126, the fractional energy release in each cubic centimeter falls off exponentially with length constant α.

Slowing Down Distribution in a Finite Block

In order to measure the slowing down distribution from a source it is customary to place the source on the axis of a long parallelepiped and measure the activity of Cd covered In foils placed along the long axis of the parallelepiped. Since In has a deep resonance at 1.44 eV, the activity of such a foil will in good part be proportional to the flux of 1.44 eV neutrons. Actually, because of higher resonances, the reading of the In foil is not quite proportional to the 1.44 eV flux; according to Hill and Roberts, at points close to a source of 30 kv neutrons in graphite, almost 40% of the activation of In is due to absorption above 1.44 eV. Farther from the source the perturbation due to higher resonances becomes less so that the mean square distance to 1.44 eV as measured by In foils is in error by much less than 40%. The theory of this experiment is a good illustration of the usefulness of the age approximation, and we give the details in the following paragraphs.

Suppose a monoenergetic unit source is placed at the point $x = 0$, $y = 0$, $z = 0$ in an infinitely long moderating prism of sides $2a$. The slowing down density satisfies

$$\Delta q(x,y,z,\tau) = \frac{\partial q(x,y,z,\tau)}{\partial \tau} \tag{3.128}$$

$$\lim_{\tau \to 0} q(x,y,z,\tau) = \partial(x,y,z) \tag{3.129}$$

where we have assumed the long direction is along z. The boundary conditions may be taken with sufficient accuracy (provided the width of the block is much larger than the mean free path).

$$q \equiv 0 \quad \text{on the extrapolated boundary} \tag{3.130}$$

the extrapolated boundary being the geometric boundary a′ augmented by the extrapolation distance 0.71 λ_{tr}. We denote a′ + .71 λ_{tr} by a. It is convenient to assume λ_{tr} independent of energy; again this is an unimportant assumption provided the block dimension is large compared to a mean free path.

The solution of Eq. 3.88 which satisfies the boundary conditions is

$$q(x,y,z,\tau) = \frac{1}{a^2} \sum_{m,n} \cos B_m x \cos B_n y \; e^{-[B_m^2+B_n^2]\tau} \frac{e^{-z^2/4\tau}}{(4\pi\tau)^{1/2}} \tag{3.131}$$

where

$$B_m = (2m+1)\frac{\pi}{2a}, \; B_n = (2n+1)\frac{\pi}{2a} \tag{3.132}$$

The sine solution is not used because of the symmetry of the source distribution.

The shape of the distribution along the z-direction is the same as from an infinite plane. As the neutrons age (τ increases) the intensity of the distribution falls because of the exponential factor. This factor accounts for leakage out of the block. Its dependence of τ arises from the circumstance that neutrons with large τ must have diffused for a relatively long time and therefore must have had a good chance to leak out of the sides. The magnitude of the leakage is determined by the ratio τ/a^2.

The distribution (Eq. 3.131) is represented as a sum of characteristic functions. The slowing down density can, of course, also be computed by observing that the neutron distribution from a point source in a finite block can be viewed as the superposition of distributions from point sources and sinks appropriately distributed in an infinite medium. The mathematical relation between the source wise and characteristic function representations of the distributions is established by means of the Poisson summation formula (Courant-Hilbert, Methoden der Mathematischen Physik, p. 65):

$$\sum_m \sum_n \varphi(m,n) = \sum_\lambda \sum_\mu \iint_{-\infty}^{\infty} \varphi(u,v)\, e^{-2\pi i(\lambda u+\mu v)}\, du\, dv \tag{3.133}$$

Upon applying this transformation to the series (3.131) with

$\varphi(m,n) = \cos B_m x \cos B_n y$

we obtain

$$q(x,y,z,\tau) = \frac{1}{(4\pi\tau)^{3/2}} \sum_{-\infty}^{\infty}\sum (a)^{\lambda+u} \exp - [(x-\lambda a)^2 + (y-\mu a)^2 + z^2]/4\tau \qquad (3.134)$$

Each term in Eq. 3.134 represents a source or sink of unit strength situated at the point (λa, μa, 0). The source wise representation of the slowing down distribution converges better than the characteristic function representation at points close to the source; at points far from the source the characteristic function form is the better converging.

Measurement of Slowing Down Length

The second moment of the distribution (3.131) is

$$\overline{z^2} = \frac{\int_0^\infty q(x,y,z,\tau) z^2 dz}{\int_0^\infty q(x,y,z,\tau)\, dz} = 2\tau$$

that is, the second moment in a finite block is the same as in an infinite block. Hence foil measurements in a block of finite width yield the same second moment as measurements in an infinite medium. This result is independent of the relative importance of the various harmonics contained in Eq. 3.131 and holds provided only that the distribution is strictly Gaussian.

Most neutron sources are not monoenergetic, nor is the slowing down intrinsically Gaussian. For both these reasons the mean square distance measured in a finite block is not strictly the same as the mean square distance in an infinite system. For example, if the energy distribution of the source is $f(\tau')d\tau'$, then

$$q(x,y,z,\tau) = \frac{1}{a^2} \sum_{m,n} \cos B_m x \cos B_n y \int_0^\infty \frac{e^{-[B_m^2+B_n^2](\tau-\tau)} f(\tau')\, e^{-z^2/4(\tau-\tau')}\, d\tau'}{[4\pi(\tau-\tau')]^{1/2}}$$

The second moment of this distribution along the z axis (x = y = 0) is

$$\overline{z^2} = \frac{\displaystyle\sum_{m,n} \int_0^\infty\int_0^\infty \frac{e^{-[B_m^2+B_n^2](\tau-\tau')} f(\tau') z^2\, e^{-z^2/4(\tau-\tau')}}{[4\pi(\tau-\tau')]^{1/2}}\, dz\, d\tau'}{\displaystyle\sum_{m,n} \int_0^\infty\int_0^\infty \frac{e^{-[B_m^2+B_n^2](\tau-\tau')} f(\tau') e^{-z^2/4(\tau-\tau')}}{[4\pi(\tau-\tau')]^{1/2}}\, dz\, d\tau'} \qquad (3.135)$$

In general this second moment will differ from the second moment $\overline{z^2_\infty}$, measured in an infinite medium:

$$\overline{z^2_\infty} = \frac{\int_0^\infty \int_0^\infty f(\tau')\, z^2\, \frac{e^{-z^2/4(\tau-\tau')}}{[4\pi(\tau-\tau')]^{1/2}}\, dz\, d\tau'}{\int_0^\infty \int_0^\infty f(\tau')\, \frac{e^{-z^2/4(\tau-\tau')}}{4\pi(\tau-\tau')}\, dz\, d\tau'} \tag{3.136}$$

Corrections must therefore be made to the observed infinite system $\overline{z^2}$ in order to obtain the true infinite system $\overline{z^2_\infty}$. It is possible to compute these corrections for a completely general kernel and this will be done in the remainder of this section.*

The corrections will be made by observing that the neutron distribution in a finite block can be considered as the sum total of effects from a suitable distribution of positive and negative sources in an infinite medium, provided as we shall assume, the extrapolation distance can be neglected compared to the block size, or is independent of neutron energy.

Now a point source at the center of the z = 0 plane in a long block of sides 2a is equivalent to a sequence of positive and negative sources spaced at intervals of 2a in the z = 0 plane. Such a sequence can be represented as

$$\partial(x)\ \partial(\gamma,\alpha) = S(x,y)\ \partial(z) = \partial(z) \sum_{m,n} \cos B_m x \cos B_n y \tag{3.137}$$

where x and y are allowed to have any value from $-\infty$ to $+\infty$.

We consider the function

$$q(x,y,z) = \int_{-\infty}^{\infty}\!\!\int S(x',y')\, P[\sqrt{(x-x)^2 + (y-y)^2 + z^2}]\, dx'dy' = \sum_{m,n} \int_{-\infty}^{\infty}\!\!\int \cos B_m x' \cos B_n y'$$

$$\times\, P[\sqrt{(x-x')^2 + (y-y')^2 + z^2}]\, dx'dy' = \sum_{m,n} \cos B_m x \cos B_n y\ \overline{P}(B_m^2 + B_n^2, z) \tag{3.138}$$

where $\overline{P}(B_m^2 + B_n^2, z)$ is the two dimensional Fourier transform of the point slowing down kernel, $P(\underline{r})$:

$$\overline{P}(B_m^2 + B_n^2, z) = \int_{-\infty}^{\infty}\!\!\int e^{-i}[B_m \xi + B_n \eta]\ P[\sqrt{\xi^2 + \eta^2 + z^2}]\, d\xi d\eta. \tag{3.139}$$

The function q(x,y,z) can be viewed as the slowing down density in an infinite medium in which the infinite array of positive and negative sources defined by Eq. 3.137 is situated. Since according to Eq. 3.138 q(x,y,z) vanishes on the boundary of the block, it can also be viewed as the slowing down density in the finite system due to a single point source at x = y = z = 0, provided the extrapolation distance is energy independent. A range measurement results in the observed $2k^{th}$ moment $\overline{z^{2k}}$.

*M. E. Rose and A. M. Weinberg, MonP-297.

$$\overline{z^{2k}} = \frac{\int_{-\infty}^{\infty} z^{2k} q(o,o,z)dz}{\int_{-\infty}^{\infty} q(o,o,z)dx}$$

$$= \frac{\sum_{m,n} \iiint_{-\infty}^{\infty} z^{2k} \cos B_m x' \cos B_n y' P(\sqrt{x'^2 + y'^2 + z^2})\, dx'dy'dz'}{\sum_{m,n} \iiint_{-\infty}^{\infty} \cos B_m x' \cos B_n y' P(\sqrt{x'^2 + y'^2 + z^2})\, dx'dy'dz'} \qquad (3.140)$$

We now show how the moments in an infinite system can be expressed in terms of the observed moments in the finite system. Since $P(x,y,z)$ is an even function of x,y,z, we can replace $\cos B_m x' \cos B_n y'$ by $\cos(B_m x + B_n y)$ in Eq. 3.140. Now

$$\cos(B_m x + B_n y) = \sum_{\nu} \sum_{s=0}^{\nu} \frac{(-1)^{\nu}}{2\nu!}\, {}_{2\nu}C_{2s}\, B_m^{2(\nu-s)} B_n^{2s} x^{2(\nu-s)} y^{2s} \qquad (3.141)$$

where ${}_{2\nu}C_{2s}$ is the binomial coefficient

$${}_{2\nu}C_{2s} = \frac{2\nu!}{[2(\nu)-s)]!\ (2s)!}$$

Upon substituting (3.141) into (3.140) we find

$$\overline{z^{2k}} = \frac{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} \frac{(-1)^{\nu}}{[2(\nu-s)]!\ (2s)!} B_m^{2(\nu-s)} B_n^{2s} \iiint_{-\infty}^{\infty} x^{2(\nu-s)} y^{2s} z^{2k} P(r)dxdydz}{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} \frac{(-1)^{\nu}}{[2(\nu-s)]!\ (2s)!} B_m^{2(\nu-s)} B_n^{2s} \iiint_{-\infty}^{\infty} x^{2(\nu-s)} y^{2s} P(r)dxdydz} \qquad (3.142)$$

The integrals which appear in Eq. 3.142 are of the form

$$\overline{x_\infty^{2i} y_\infty^{2j} z_\infty^{2L}} = \iiint_{-\infty}^{\infty} x^{2i} y^{2j} z^{2L}\ P(r)\ dxdydz$$

and can be evaluated by shifting to polar coordinates. Thus

$$\overline{x_\infty^{2i} y_\infty^{2j} z_\infty^{2L}} = \frac{(2i)!\ (2j)!\ (2L)!\ (i+j+L)!}{i!\ j!\ L!\ (2i+2j+2L)!}\ \overline{z_\infty^{2i+2j+2L}} \qquad (3.143)$$

Upon substituting Eq. 3.143 into Eq. 3.142 we obtain

$$\overline{z^{2k}} = \frac{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} (-1)^{\nu} B_m^{2(\nu-s)} B_n^{2s} \frac{(2k)!\ (\nu+k)!\ \overline{z_\infty^{2k(\nu+1)}}}{s!\ (\nu-s)\ k!\ [2(\nu+k)]!}}{\sum_m \sum_n \sum_\nu \sum_{s=0}^{\nu} (-1)^{\nu} B_m^{2(\nu-s)} B_n^{2s} \frac{\nu!\ \overline{z_\infty^{2\nu}}}{s!\ (\nu-s)!\ (2\nu)!}} -$$

which is an infinite system of linear equations relating the observed moments $\overline{z^{2k}}$ to the infinite system moments $\overline{z_\infty^{2k}}$. The system can be solved for each $\overline{z_\infty^{2k}}$

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in terms of the $\overline{z^{2k}}$ by successive approximations, in which, at each stage of the approximation only a finite number of equations and unknowns are used. Such a process will converge well if the block dimension is large compared to the slowing down range.

If the source instead of being concentrated in a point is distributed over the z = 0 plane like $\cos B_0 x \cos B_0 y$, only the term m = n = 0 appears in Eq. 3.144. The infinite second moment can then be expressed explicitly in terms of the measured finite system moments:

$$\overline{z_\infty^2} = \frac{\overline{z^2} + \frac{1}{3} B_0^2\overline{z^4} + \frac{1}{30} B_0^4\overline{z^6}\ldots}{1 + B_0^2\overline{z^2} + \frac{1}{6} B_0^4\overline{z^4} + \frac{1}{90} B_0^6\overline{z^6}\ldots} \qquad (3.145)$$

and this expression gives the correction for converting $\overline{z^2}$ into $\overline{z_\infty^2}$. Equation 3.145 is of practical importance since measurement of fission neutron ranges are sometimes performed by using the thermal neutrons from a thermal column which are distributed like $\cos B_0 x \cos B_0 y$ to produce fissions in a flat plate of fissionable material. The fission neutrons in such an arrangement will be distributed also as $\cos B_0 x \cos B_0 y$.