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# Enhancement of Particle-Wave Energy Exchange by Resonance Sweeping

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## Abstract

It is shown that as the resonance condition of the particle-wave interaction is varied adiabatically, that the particles trapped in the wave will form phase space holes or clumps that can enhance the particle-wave energy exchange. This mechanism can cause much larger saturation levels of instabilities, and even allow the free energy associated with instability, to be tapped in a system that is linearly stable due to background dissipation.

**key words:** *resonance, sweeping, hole, clump, adiabatic, free energy, wave momentum*

# I. INTRODUCTION

There are many cases in plasma physics where the resonant interaction of particles and waves determine interesting physical phenomena. A common example arises when a coherent mode, established by the interaction of the electromagnetic field and the majority of plasma particles, is destabilized through a particle-wave resonance with a minority component that is supplying "free energy" to feed the wave. Examples vary from the basic bump-on-tail instability[1–3] arising in plasma oscillations to the instability of a fusion producing plasma in a torus where the Toroidal Alfvén Eigenmode[4–7] can be destabilized by alpha particles that tap the universal instability drive. Another type of instability is exemplified by the so-called fish-bone instability,[8–10] where the minority species is needed to both establish the wave, and destabilize it due to a resonant energy exchange mechanism. Other examples of the use of particle resonances occur in applications to current drive[11] and energy "channelling".[12]

In this note we wish to point out a mechanism first reported in Ref. 13, whereby the power transfer of particles to waves can be enhanced from the prediction of linear theory through the adiabatic transport of resonant particles trapped in a coherent wave as the resonance location in phase space is continuously varied. Another study of such effects has been discussed by Mynick and Pomphrey,[14] where they primarily addressed how particles can be extracted by frequency sweeping and they noted that energy can also be extracted by this method. Here we show how to apply the results of Ref. 13 to determine the field amplitude and energy conversion level as a result of sweeping. By scanning frequency sufficiently slowly, trapped particles can remain trapped in a wave as the resonance condition changes and thereby phase space gradients build-up as the trapped particles, with their specified distribution, convect into other phase space regions where the surrounding distribution is different. We shall show that these gradients, which, depending on the system and the direction of the frequency scan, take either the form of phase space "holes"[15] (where the trapped distribution is depressed from neighboring regions) or phase space "clumps"[16] (where the trapped distribution is enhanced from neighboring regions) enhance the energy transfer rate of the resonant particles to the wave. This phenomenon has the effect of allowing for much larger saturation levels than otherwise would be expected in an unstable system. In a system linearly stabilized by background dissipation, the enhanced energy exchange associated with this mechanism, allows the wave energy to spontaneously grow from an imposed low level seed perturbation. This phenomena allows energy channelling to be more efficient than otherwise, as background dissipation no longer sets a stringent bound for tapping the particle free energy.

## II. POWER TRANSFER BY ADIABATIC RESONANCE SWEEPING

In order to tap the free energy reservoir of weak instabilities it is normally necessary for the linear instability drive to overcome dissipation from mechanisms related to the background plasma. The evolution of the wave energy can be described as the rate of change of wave energy $W$, being equal to the power extracted from the linear drive, minus the power absorbed by the background plasma through dissipation. In linear theory all the terms of such an equation are proportional to the square of the electric field, $\mathbf{E}$, of the wave. In this discussion it will be convenient to use the mean trapping frequency, $\varpi_B$, for particles deeply trapped in a finite amplitude wave as one can express in a universal way, the response of any physical system with weak instability in terms of $\varpi_B$. The trapping frequency of a particle, $\omega_B$, is proportional to $|\mathbf{E}|^{1/2}$. The kinetic particles driving the instability have a "free energy" $W_F$, in a volume $V$ occupied by a single mode, that in principle can be transferred to the mode. Roughly, the free energy is the energy that has to be extracted from the particles to make the kinetic distribution of the particles be stabilizing, and the free energy is generally comparable to the kinetic energy of the particles. It can be shown that the relation between wave energy, free energy, and bounce frequency scales roughly as

$$W \approx \frac{W_F}{\gamma_L} \frac{\varpi_B^4}{\overline{\Omega}^3} \tag{1}$$

where $\overline{\Omega}$ is the spread of the frequency resonance function over the particle distribution function (for a particle in resonance $\Omega = 0$), and $\gamma_L$ is the linear growth rate. We will primarily use the electrostatic one-dimensional bump-on-tail problem as a paradigm, as for a given mode, $\omega_B$ needs to be determined only at the one point in velocity space where there is resonance. However, our results are quite general if a mean value of $\omega_B$ is taken. In the bump-on-tail problem, the resonance is $\Omega = \omega - kv$, with $\omega$ the mode frequency, $k$ the mode wavenumber, $v$ the particle velocity, and the bounce frequency is given by $\omega_B = (\frac{e}{m} k\widehat{E})^{1/2}$ with $\widehat{E}$ the electric field amplitude.

Now the equation for the wave energy evolution can be written as

$$\frac{\partial}{\partial t} W = 2\gamma_L W - 2\gamma_d W \tag{2}$$

with $\gamma_d$ the damping rate. Upon substituting Eq. (1) in Eq. (2), this relation becomes

$$\frac{\partial}{\partial t} \varpi_B^4 = 2\gamma_L \varpi_B^4 - 2\gamma_d \varpi_B^4. \tag{3}$$

It has been shown[17–21] that under many conditions a single mode saturates at a level where $\varpi_B \sim \gamma = \gamma_L - \gamma_d$. The free energy released by particles to the wave, $\Delta W$, is then found from Eq. (1) to be

$$\Delta W \simeq W_F \left(\frac{\gamma}{\overline{\Omega}}\right)^3 \frac{\gamma}{\gamma_L}. \tag{4}$$

This is a rather low level of conversion of free energy to wave energy.

We now note that we can convert considerably more free energy if we can slowly change the resonance position in time. This problem has already been analyzed analytically in Ref. 13, where it was assumed that the destabilizing particles were slowing down due to drag, and that diffusive processes such as pitch angle scattering, were unimportant. In fact it turns out that the treatment of the problem is identical whether the resonance condition (recall $\Omega = \omega - kv$ for the electrostatic problem) for a particle changes, due to drag (then $\frac{d\Omega}{dt} = -k\frac{dv}{dt} = \nu kv = \nu\omega$ where $\frac{dv}{dt} = -\nu v$), or due to the mode frequency changing in time (for which $\frac{d\Omega}{dt} = \frac{d\omega}{dt} = \nu\omega$, where now $\nu = \frac{1}{\omega}\frac{d\omega}{dt}$). The analysis shows that if $\frac{d\Omega}{dt} \ll \omega_B^2$, there will be an adiabatic invariant that causes the particles trapped in the wave field to remain trapped as the resonance changes. This means that as the resonance is swept, particles that are originally trapped at a phase velocity $\frac{\omega(0)}{k} = v_{ph}(0)$, and having a distribution weight of $f_0\Big(v_{ph}(0)\Big)$, keep the same distribution weight as the particles adiabatically track the resonance. Thus when the phase velocity becomes $\frac{\omega(t)}{k} = v_{ph}(t)$, the weight of the deeply trapped particles will still be $f_0\Big(v_{ph}(0)\Big) \neq f_0\Big(v_{ph}(t)\Big)$. Thus, there will be a rapid change of $f$ near the instantaneous phase velocity of the wave. The power transfer arising from this rapid change in $f$ can be readily calculated.

A quantitative expression for the power transfer arises from a straightforward physical picture. We first consider the simplest physical system, that of the one-dimensional bump-on-tail instability. In this case the resonance function is $\Omega = \omega - kv$, where we take $\omega$ to be time dependent, and $k$ a constant wavenumber. The electrostatic field is taken as $E = \widehat{E}\,\sin(\psi)$, with $\psi = kx - \int_0^t \omega dt$, and we define the quantity $\nu = \dot{\omega}/\omega$.

To calculate the power transfer we first calculate the momentum, $\Delta M$, that is removed in a time $\Delta t$, from the free energy reservoir, as the frequency is being swept. This momentum is transferred to the wave. From basic principles, the wave energy, $\Delta E$, added to the wave is $\Delta E = \Delta M\,\omega/k$. The power transfer, $P$, is then $P = \frac{\Delta M}{\Delta t}\frac{\omega}{k}$.

Let us compare the phase space contours of the distribution function (of the particles constituting the free energy reservoir) at a time $t_0$ and a time $t_0 + \Delta t$, where we take $\frac{\widehat{E}}{\frac{d}{dt}\widehat{E}} \gg \Delta t \gg \frac{\omega_B}{\nu\omega}$, with $\omega_B$ is the bounce frequency of the particles trapped in the wave. The phase space plot is shown in Fig. 1. We assume $\nu$, $\frac{\partial f(0)}{\partial v} > 0$. Note that the $f$-values of the distribution function inside the separatrices is then substantially below the $f$-value of the distribution function outside the separatrices. Outside the separatrix the distribution is essentially $f_0\left(\frac{\omega}{k}\right)$ with $f_0$ the unperturbed distribution function. Thus it is clear that the kinetic distribution at $t = t_0 + \Delta t$, with its hole at higher speeds, has less energy than at $t = t_0$, with the energy difference being converted to wave energy. To quantify the energy transfer we note that inside the separatrix the distribution function is a function of the adiabatic invariant,

$$J = \int_{-\psi_0}^{\psi_0} d\psi \left(v - \frac{\omega(t)}{k}\right) = \sqrt{2} \int_{-\psi_0}^{\psi_0} d\psi \left[\varepsilon + \frac{e\widehat{E}(t)}{mk} \cos\psi\right]^{1/2} \tag{5}$$

with the particle energy $\varepsilon$ in the wave frame, held constant during the $\psi$-integration, and $\cos\psi_0 = -mk\,\varepsilon/e\,\widehat{E}(t)$. We have assumed $\nu \ll \omega_B^2/\omega$, so that the adiabatic invariant is conserved as the frequency is swept. Note that $v - \omega(t)/k \equiv \delta v(\psi, J)$. For a given $J$, the value of $f$ is the value the passing particles had at time $t'$ when these passing particles were just skimming the separatrix. For these particles, $\varepsilon = e\widehat{E}(t')/mk$ and $J \equiv J(t') = 8\left(\frac{e\widehat{E}(t')}{mk}\right)^{1/2}$. Thus we have

$$f\left[J = 8\left(\frac{e\widehat{E}(t')}{mk}\right)^{1/2}\right] = f_0\left(\frac{\omega(t')}{k}\right). \tag{6}$$

Now the difference of momentum (averaged over a spatial period) of the free energy reservoir at $t + \Delta t$ and $t$ is given by

$$\Delta M = \frac{k}{2\pi} \int_{-\pi/k}^{\pi/k} dx \int_{-\infty}^{\infty} dv\, mv \left[f(t+\Delta t) - f(t)\right]$$

$$= m \int_{-\pi}^{\pi} \frac{d\psi}{2\pi} \left[\int_{\frac{\omega}{k}(t_0+\Delta t)-\delta v(J,\psi)}^{\frac{\omega}{k}(t_0+\Delta t)+\delta v(J,\psi)} dv\, v \left(f(t+\Delta t) - f_0\right) - \int_{\frac{\omega}{k}(t_0)-\delta v(J,\psi)}^{\frac{\omega}{k}(t_0)+\delta v(J,\psi)} dv\, v \left(f(t+\Delta t) - f_0\right)\right]. \tag{7}$$

At time $t_0 + \Delta t$, we have $v = \frac{\omega(t_0+\Delta t)}{k} + \delta v \doteq \frac{\omega(t_0)}{k} + \nu \Delta t \frac{\omega(t_0)}{k} + \delta v$, and at time $t$ we have $v = \frac{\omega(t_0)}{k} + \delta v$. Note that $f(t+\Delta t) - f_0 = f(t) - f_0$, because $f(t) = f(J)$ is constant and there is a negligible change of the passing particle distributions, $f_0$, due to the small difference of phase velocity, $(\nu\Delta t\omega/k)$. We then find

$$\Delta M = \frac{m\omega\nu\Delta t}{k}\int_0^{J(t)} \frac{dJ}{\pi}\Big(f(J) - f_0\Big).$$

The power (per unit length) transferred to the wave, $P = -\frac{\omega}{k}\frac{\Delta M}{\Delta t}$, is then

$$P = \frac{m\omega^2}{k^2}\frac{\nu}{\pi}\int_0^{J(t)} dJ\Big(f_0 - f(J))\Big) = \frac{m\omega^2}{k^2}\frac{\nu}{\pi}\Big(J(t)f_0 - \int_0^{J(t)} dJ f(J)\Big) < \frac{8m\omega^2}{\pi k^3}\nu\omega_B f_0. \quad (8)$$

The inequality follows from our assumptions that $\frac{\partial f_0}{\partial v}$, $\frac{\omega}{k}$, and $\nu$ are greater than zero.

If we assume $\frac{\omega(t)}{k} - \frac{\omega(0)}{k} \ll f_0 \Big/ \frac{\partial f}{\partial v}$, we can somewhat simplify Eq. (8). We use the approximation,

$$f(J) \doteq f_0\left(\frac{\omega(t)}{k}\right) + \left(\frac{\omega(t'(J))}{k} - \frac{\omega(t)}{k}\right)\frac{\partial f_0(\omega(t)/k)}{\partial v}. \quad (9)$$

Then we substitute into Eq. (8) to find

$$P = \frac{8m}{\pi}\frac{\omega^2}{k^4}\nu\frac{\partial f_0(\frac{\omega}{k})}{\partial v}\left(\omega_B(t)\Big(\omega(t) - \omega(0)\Big) - \int_0^t dt'\frac{\partial\omega_B(t')}{\partial t'}\Big(\omega(t') - \omega(0)\Big)\right)$$

$$\frac{8m}{\pi}\frac{\omega^2\,(\omega(t) - \omega_0)}{k^4}\nu\frac{\partial f_0(\frac{\omega}{k})}{\partial v}\omega_B(t) \approx \frac{\omega_B W_F \nu\omega^2\Big(\omega(t) - \omega(0)\Big)}{\overline{\Omega}^4} \quad (10)$$

where we have assumed $\overline{(v - \overline{v})^2} \approx \overline{v}^2$.

The above formula for $P$ also applies if $\nu < 0$. As $\omega(t) - \omega(0) \propto \nu$, the power transfer, $P$, is independent of the sign of $\nu$. Now instead of holes forming during adiabatic sweeping, clumps form.

Using Eq. (1) for the wave energy, and (10) for the power transfer, we find that the change in $\omega_B$ due to the adiabatic frequency shift and dissipation is

$$\frac{\partial\omega_B^4}{\partial t} \approx \nu\omega_B\omega^2(\omega - \omega(0))\gamma_L\Big/\overline{\Omega} - \gamma_d\,\omega_B^4. \quad (11)$$

Now for simplicity we will assume $\overline{\Omega} \approx \omega$. Then a rough integration of this equation over a time $t \sim 1/\nu$, shows that

$$\omega_B \approx (\gamma_L \omega^2)^{1/3}. \tag{12}$$

Now using Eq. (1), the free energy release is found to be

$$\Delta W = W_F \left(\frac{\gamma_L}{\omega}\right)^{1/3} \tag{13}$$

To achieve these levels of $\omega_B$ in a stable system where $\gamma_d > \gamma_L$, one must either increase $|\nu|$ in time as $\omega_B$ increases or initially impose a sufficiently large value of $\omega_B$ with an external source. To have growth in $\omega_B$, $\nu$ must be large enough so that the enhanced power transfer will be greater than the power dissipated and small enough that the adiabaticity condition, $\nu < \omega_B^2/\omega$, is not violated. Thus with an initially imposed $\omega_B = \omega_{B0}$, we require in the right-hand side of Eq. (11), that the drive term be larger than the dissipation term, which means

$$\nu \gamma_L \omega_B \omega \int_0^t dt \nu \omega > \gamma_d \omega_B^4. \tag{14}$$

Initially $\nu$ can be no larger than $\omega_{B0}^2/\omega$. If $\nu$ is constant, the value of $\omega_B$, that can be achieved in a time $t \sim 1/\nu$ is then found to be

$$\omega_B \simeq \min\left[(\gamma_L \omega^2)^{1/3},\ \left(\omega_{B0}^2 \omega \gamma_L/\gamma_d\right)^{1/3}\right]. \tag{15}$$

Thus, if $\omega_{B0} < (\gamma_d \omega)^{1/2}$, the level of $\omega_B$ that can be achieved is lower than the potentially achievable level of $(\gamma_L \omega^2)^{1/3}$.

When $\omega_{B0} < (\gamma_d \omega)^{1/2}$, the level, $\omega_B \simeq (\gamma_L \omega^2)^{1/3}$, can still be achieved if $\nu$ changes with time. An optimal strategy to achieve maximum energy conversion is to set $\nu \approx \omega_B^2/\omega$. Then, after a short interval of time, $t \sim \frac{\gamma_d}{\gamma_L \omega_{B0}}$, the left-hand side of Eq. (14) exceeds the right hand, and $\omega_B$ will grow according to Eq. (11). As $\omega_B$ increases, the drive term continues to increase faster than the dissipation term, because the driven term scales as $\omega_B^5$ when $\nu \approx \omega_B^2/\omega$, while the dissipation term scales as $\omega_B^4$. The integration of Eq. (11), over a time in which the change in $\omega$ is comparable to its original value, will then give $\omega_B \approx (\gamma_L \omega^2)^{1/3}$. Note that integration for a larger time will not be effective when $\overline{\Omega} \approx \omega$, since then the resonance $\Omega = 0$ would correspond to a phase space position not occupied by the passing particle distribution. Note that when $\nu > \gamma_d$, $\nu$ need no longer change in time, and $\omega_B \simeq (\gamma_L \omega^2)^{1/3}$ can still be achieved. However, if the final value of $\nu$ is less than $\gamma_d$, dissipation sets a limit on $\omega_B$. The final value can be expressed as

$$\omega_B \approx (\omega^2 \gamma_L)^{1/3} \min\left(1, \frac{\nu}{\gamma_d}\right). \tag{16}$$

When the background damping determines the level of $\omega_B$, more energy dissipates to the background plasma (this is the channelling mechanism) than instantaneously exists in the wave. If $\nu/\gamma_d < 1$ and the wave exists at the level $\omega_B \approx (\omega^2/\gamma_L)^{1/3}\nu/\gamma_d$ for a time $t \simeq 1/\nu$, and the amount of energy $\Delta W_{\mathrm{ch}}$, that is channelled to the background plasma, $\propto \gamma_d \int_0^{\nu^{-1}} dt\, \omega_B^4$, is

$$\Delta W_{\mathrm{ch}} \simeq W_F \left(\frac{\gamma_L}{\omega} \frac{\nu}{\gamma_d}\right)^{1/3}. \tag{17}$$

When $\nu > \gamma_d$, a free energy release, $W_F(\frac{\gamma_L}{\omega})^{1/3}$, is achieved after the completion of the frequency scan. This energy can either be allowed to be absorbed by the background plasma through dissipative processes, or directly converted to a grid with external antennae circuitry.

## III. EFFECT OF VELOCITY DIFFUSION

Another important physical process is particle scattering from either small angle collisions or velocity space diffusion arising from external heating. This process can make the achievement of an adiabatic scanning process more difficult. Straightforward dimensional arguments show that diffusion processes cause trapped particles to escape from the trapping region at a rate given by $\nu_{\mathrm{eff}} \simeq \nu_s \omega^2/\omega_B^2$, where $\nu_s$ is the rate of relaxation of the overall velocity distribution. Then, if we scan the frequency for a time greater than $1/\nu_{\mathrm{eff}}$, the change in the distribution, $\Delta f$, in the trapping region, is

$$\Delta f \approx \frac{\partial f}{\partial v}\frac{\nu\omega}{k}\frac{1}{\nu_{\text{eff}}}. \tag{18}$$

As a result, in Eq. (11), if the scan time is greater than $1/\nu_{\text{eff}}$, we need to replace $\omega(t)-\omega(0)$ by $\nu\omega/\nu_{\text{eff}} = \nu\omega_B^2/\omega\nu_s$. This result has been obtained rigorously in Ref. 13. Analytically we can model arbitrary $\nu_{\text{eff}}$ if, in Eq. (11) we replace

$$\left(\omega(t)-\omega(0)\right) \rightarrow \int_0^t dt'\nu\omega \exp\left(-\int_0^{t'} dt''\nu_{\text{eff}}\right). \tag{19}$$

When $\frac{1}{\omega_B^2}\frac{\partial\omega_B}{\partial t} \ll \nu_{\text{eff}}$, we see that both limits are recovered. Thus, when $\nu_{\text{eff}}t > 1$, Eq. (11) changes to

$$\frac{\partial\omega_B^4}{\partial t} = \frac{\nu^2}{\nu_s}\omega_B^3\gamma_L - \gamma_d\,\omega_B^4. \tag{20}$$

The diffusion regime defined by Eq. (20) sets more stringent conditions to start-up. We will see that if $(\frac{\gamma_d}{\omega})^{1/2}\gamma_L > \nu_s > \gamma_d^2/\omega$, a minimum perturbation of $\omega_{b0} > (\gamma_d\omega)^{1/2}$ is required at start-up in order to be able to amplify to a level of $\omega_B \simeq (\omega^2\gamma_L)^{1/3}$. Further, if $\nu_s > (\frac{\gamma_d}{\omega})^{1/2}\gamma_L$, we will not be able to substantially amplify any initial perturbation.

To demonstrate these contentions, suppose $\nu_s$ satisfies the condition $\nu_s\omega^2/\omega_{B0}^2 > \frac{\omega_{B0}^2}{\omega} > \nu$, so that we are guaranteed to be in the diffusion regime at start-up. Then in order to have growth in $\omega_B$, we require $\frac{\omega_{B0}^7\gamma_L}{\omega^4\nu_s} > \nu^2\omega_{B0}^3\gamma_L/\nu_s > \gamma_d\omega_{B0}^4$. Thus growth of $\omega_B$ is always possible if $\nu_s < \omega_{B0}^3\gamma_L/\omega^2\gamma_d$. Assuming this condition well satisfied, then $\omega_B$ changes by a substantial amount (i.e. $\omega_B(t)-\omega_B(0) \gtrsim \omega_{B0}$) when $\nu^2 t\gamma_L/\nu_s\omega_{B0} \approx 1$, and the system can enter the diffusionless regime. We wish the inequality to arise when $\nu t \ll 1$ and $\nu \lesssim \omega_{B0}^2/\omega$, so that the optimum, $\omega_B \sim (\omega^2\gamma_L)^{1/3}$, can be obtained. Thus we wish to have the following inequalities satisfied

$$1 \gg \nu t = \frac{\nu_s\omega_{B0}}{\nu\gamma_L} > \frac{\nu_s\omega}{\omega_{B0}\gamma_L},$$

or, equivalently, $\nu_s < \omega_{B0}\gamma_L/\omega$. Thus the condition to be initially in the diffusion regime, but still have the ability to amplify $\omega_B$ into the diffusionless regime, is

$$\frac{\omega_{B0}^3\gamma_L}{\omega^2\gamma_d} > \frac{\omega_{B0}\gamma_L}{\omega} > \nu_s > \frac{\omega_{B0}^4}{\omega^3}. \tag{21}$$

The first two inequalities on the left establish that we need $\omega_{B0} > (\gamma_d\omega)^{1/2}$. Then $\nu_s$ lies in the regime

$$\left(\frac{\gamma_d}{\omega}\right)^{1/2}\gamma_L > \nu_s > \frac{\gamma_d^2}{\omega}.$$

Note that if we do not satisfy the left-hand inequality, the initial perturbation cannot be substantially amplified. If the right-hand inequality is violated, we can choose start-up parameters that are in the diffusionless regime.

We now note that in unstable systems $\omega_B$ can be amplified above the natural level, $\omega_B \sim \gamma_L$, when the frequency is scanned nonadiabatically ($\nu > \omega_B^2/\omega$). However, the potential amplification level is still lower than the adiabatic case.

In the nonadiabatic regime, new particles enter the region of resonance at the rate

$$\frac{d\Delta N}{dt} \simeq \frac{\nu\omega}{\omega_B}\Delta N \tag{22}$$

where $\Delta N$ is the number of trapped particles. If $\frac{\nu\omega}{\omega_B} > \omega_B$, the trapped distribution does not have a chance to flatten, and the wave will continue to grow. On the other hand, the change of the frequency is comparable to its original value in a time $t \sim 1/\nu$. Thus, in order to have at least one growth time of amplification in the nonadiabatic regime, we need

$$\gamma_L > \nu > \omega_B^2/\omega. \tag{23}$$

Thus, the level of $\omega_B$ that can be reached in a nonadiabatic scan is $\omega_B \lesssim (\omega\gamma_L)^{1/2}$, which is a lower level than is in principle achievable with an adiabatic scan.

We can model these effects quantitatively as follows. We let the conventional growth rate satisfy the relationship

$$\gamma' = \frac{\gamma_L}{[1 + \frac{\omega_B}{3\gamma' + \nu\omega/\omega_B + \nu_{\text{eff}}}]}. \tag{24}$$

For $\omega_B = 0$ we recover linear theory. If there are no sources ($\nu = 0, \nu_{\text{eff}} = 0$), we have $\gamma \to 0$ when $\omega_B \to 3\gamma_L$, which is a result previously observed in particle simulations. In Ref. 13, it is shown the $\gamma \simeq \gamma_L \, \nu_{\text{eff}}/\omega_B$ if $\nu_{\text{eff}}/\omega_B \ll 1$, where $\nu_{\text{eff}}$ represents the rate at which particles re-enter the flattened resonance region due to diffusion. Here we have conjectured that by scanning the frequency rapidly, we have introduced a similar source as $\nu_{\text{eff}}$, which is now given by $\nu\omega/\omega_B$. We can then write a set of model equations that include all the processes we have described. In the adiabatic term we include an exponential cut-off when $\nu\omega > \omega_B^2$. The model equations are

$$\frac{d\omega_B^4}{dt} = -\gamma_d\,\omega_B^4 + \gamma'\omega_B^4 + \exp\left(-\frac{\nu\omega}{\omega_B^2}\right)\omega\nu\gamma_L\omega_B \int_0^t dt'\nu\omega\,\exp\left(-\int_0^{t'} dt''\nu_{\text{eff}}\right) + \tilde{\omega}_b^4(t)(\gamma_d - \gamma_L) \tag{25}$$

where $\tilde{\omega}_b(t)$ is included to describe an imposed external perturbation. A set of solutions to this equation is shown in Fig. 2.

## IV. GENERALIZATION TO TOROIDAL SYSTEMS

The preceding considerations also apply to more complicated systems. For example, for waves in a toroidally symmetric plasma such as a tokamak, where the perturbed amplitude varies as $\hat{A}(t)\sin(n\phi - \omega t)$, one can still show that Eqs. (1)–(4) are valid. The concept of particles trapped in a wave at a bounce frequency $\omega_B \propto \hat{A}^{1/2}$ is still valid. The resonance condition is $\Omega = \omega - n\bar{\omega}_\phi(H, P_\phi, \mu) - p\bar{\omega}_\theta(H, P_\phi, \mu)$, where $\bar{\omega}_\phi$, and $\bar{\omega}_\theta$ are, respectively, the average toroidal and poloidal drift frequencies of a particle, $n$ and $p$ are integers, $H$ is the particle energy, $P_\phi$ is the toroidal canonical angular momentum and $\mu$ is the magnetic moment. It follows from general arguments that $H' = H - \frac{\omega}{n} P_\phi$ is conserved as a particle interacts with a toroidal wave. If the wave is low frequency, $\mu$ is conserved as well. If we consider an adiabatic sweeping of the frequency of low frequency waves (i.e. $d\omega/dt \equiv \nu\omega \ll \omega_B^2$), then for a particle trapped in the wave, $P_\phi$ will change while $H'$ and $\mu$ remain constant, according to the relation

$$\frac{d\Omega}{dt} = \nu\omega - \frac{dP_\phi}{dt}\frac{\partial}{\partial P_\phi}\bigg|_{H'} (n\varpi_\phi + p\varpi_\theta) = 0. \tag{26}$$

The trapped particles will then track the resonance, and one will again generate large phase space gradients. It can be shown that the adiabatic invariant can be written as

$$J = \int_{-\psi_0}^{\psi_0} d\psi (P_\phi - P_{\phi r}) = \sqrt{2}\int_{-\psi_0}^{\psi} d\psi \left[\varepsilon + \frac{\omega_B^2 \cos\psi}{\left(|\partial\Omega/\partial P_\phi|_{H'}\right)^2}\right]^{1/2} \tag{27}$$

where

$$\Omega(P_{\phi r}) = 0, \quad \varepsilon = \frac{(P_\phi - P_{\phi r})^2}{2} - \frac{\omega_B^2 \cos\psi}{\left(\frac{\partial\Omega}{\partial P_\phi}\right)^2}. \tag{28}$$

By using similar arguments as in the one dimensional electrostatic problem, the power converted to the wave is found to be

$$P = \frac{\Delta E}{\Delta t} = \frac{\omega}{n}\frac{\Delta M_\phi}{\Delta t} \tag{29}$$

where $\Delta M_\phi$ is the toroidal component canonical momentum that is lost by the resonant particle during the time interval $\Delta t$ as the frequency is swept. We then find, in a manner analogous to the arguments given above, that

$$\begin{aligned} P &= \frac{m\omega^2}{n}\frac{\nu}{2\pi}\int \frac{d^3v\, d^3r\, \delta}{\left|\frac{\partial\Omega}{\partial P_\phi}\right|_{H'}}\Big(P_\phi - P_{\phi r}\Big)\frac{1}{J(t)}\int_0^{J(t)} dJ\Big(f_0 - f(J)\Big) \\ &\lesssim \frac{8m\omega^2\nu}{\pi n}\int \frac{\omega_B}{\left|\frac{\partial\Omega}{\partial P_\phi}\right|^2_{H'}} f_0\, d^3r\, d^3v\, \delta\left(P_\phi - P_{\phi 0}\right). \end{aligned} \tag{30}$$

This expression is a generalization of the expression obtained in Ref. 13 where the calculation was done for a plasma slab when the mode frequency is much less than the diamagnetic frequency of the kinetic component. Observe that in terms of $\varpi_B$, nearly all conclusions of the bump-on-tail problem apply to the more complicated problem. The main difference is that in more complicated systems, $\varpi_B$ is a suitable average over particles in resonance, whereas in the bump-on-tail problem there is only one point resonant in phase space.

## V. DISCUSSION

In summary we discuss our results and their implications.

(1) If the frequency changes there can be an enhanced saturation level of an unstable wave. Without a frequency change, the saturation level is given by $\omega_B \approx \gamma_L + (\frac{\gamma_L \omega^2 \nu_*}{\gamma_d})^{1/3}$ (the case $\gamma_L \doteq \gamma_d$[19] is somewhat more complicated and not discussed here). This level can be enhanced to $\omega_B \approx (\gamma_L \omega^2)^{1/3} \min(1, \frac{\nu}{\gamma_d})$ by an adiabatic scan of the frequency, where the frequency scan rate, $\nu$, satisfies $\nu < \omega_B^2/\omega$. This level implies that in a single scan, the fraction of free energy that can be converted to wave energy is $(\gamma_L/\omega)^{1/3}$. High efficiency of energy conversion of free energy to wave energy may be possible with multiple scans. When $\nu > \omega_B^2/\omega$, the response is nonadiabatic, and enhancement of $\omega_B$ is achieved over the natural level, $\omega_B \sim \gamma_L$. However, the level of $\omega_B$ that can be achieved with a nonadiabatic scan is limited to $\omega_B \sim (\omega\gamma_L)^{1/2}$, which is lower than the level of $\omega_B$ that can be achieved with an adiabatic scan.

Normally there is no reason to expect the frequency to vary during an instability of a stationary system. However, instabilities often arise in a plasma where the background properties vary with time. In the fishbone experiment, the plasma is being heated and is reaching parameters that approach marginal stability for MHD waves. In this case the frequency of the fishbone is sensitive to the closeness to marginal stability. It may be that the approach of the background plasma to marginal stability allows the frequency to change. In any event, if there exists a mechanism that allows the frequency to shift, the processes described here allows large wave amplitudes to grow and perhaps give rise to the kind of energetic particle transport that has been observed experimentally[8] and discussed in computer simulations.[22,23]

(2) For a stable system, the adiabatic response to a frequency change can allow waves to spontaneously grow from a "seed" perturbation. Our analysis shows that growth is feasible if either the initial perturbation is large enough, or if the frequency scan rate is made to increase as the mode grows. Sufficiently rapid diffusion in velocity space sets limits to obtaining large amplifications of $\omega_B$. If $(\gamma_d/\omega)^{1/2}\gamma_L > \nu_s > \gamma_d^2/\omega$, then the absolute minimum perturbation that is required to amplify to a large final $\omega_B$ level, is $\omega_{B0} \sim (\gamma_d\omega)^{1/2}$. Once a wave is created, its wave energy can be extracted by terminating the frequency scan, thereby allowing the wave to damp because of dissipation to the background plasma (this is energy channelling) or by transferring the wave energy back to a grid through appropriate phasing with an external antennae. This latter procedure is a form of direct conversion, and a further study is needed to assess the feasibility of this method. If feasible, this form of direct conversion may be applicable to a D-$He^3$ fusion system, where 15 MeV protons are a principal fusion product.

(3) We have indicated, in principle, a method where a plasma can amplify a low level signal to a level that one would not expect on the basis of linear theory. This arises because it is possible to use particle adiabaticity to create phase space holes or clumps. The scaling given here is generic to nearly every kinetic system. The only major assumption made is that resonance overlap does not occur as the wave amplitude grows. In fact, with the increased amplitudes achievable with frequency sweeping, may then allow mode overlap to occur, thereby producing global plasma transport.

We note that varying the frequency can be difficult when an external antennae excites a normal mode for which the frequency is defined by the plasma. In this case it may be necessary to change the mode frequency by changing plasma parameters in time by such methods as modulating the magnetic field or injecting particles with a pellet.

It would be interesting to exhibit the enhancement of the power extraction rate in a laboratory experiment under controlled conditions. Such experiments can be devised on a variety of different plasma systems, from a $Q$-machine to a tokamak.

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We would like to thank Dr. King-Lap Wong, Dr. Ambrosio Fasoli, and Dr. James Van Dam for interesting discussions. This work is supported by the Dept. of Energy contract #DE-FG05-80ET-53088.

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## FIGURE CAPTIONS

FIG. 1. Phase space contours of $v - \omega(t_0)/k$: (a) at time $t_0$; (b) at time $t_0 + \Delta t$. Contours of constant distribution function are plotted. If the distribution function weight at $v \approx \omega(t_0)/k$ increases with velocity in the passing region, and there is a hole in the trapped region, the system at time $t = t_0 + \Delta t$ has less kinetic energy than at time $t_0$, and this can increase the wave energy.

FIG. 2. Example of solution of Eq. (24), for the particle trapping frequency, as a function of time. The parameters for graphs (a) and (b) are chosen so that $\gamma_L = 1/2\,\gamma_d$, and $\nu = .5\omega_B^2/\omega$. Graph (c) is for the nonadiabatic case $\nu = 5\omega_B^2/\omega$. On each graph the different curves are for different normalized scattering frequencies, with $\nu_s = \omega_B^3(0)\gamma_L\lambda/\omega^2\gamma_d$. The applied frequency $\tilde{\omega}_B$ is $\gamma_L$ in curve (a) and (c), and in curve (b), $\tilde{\omega}_B = 5\gamma_L$.

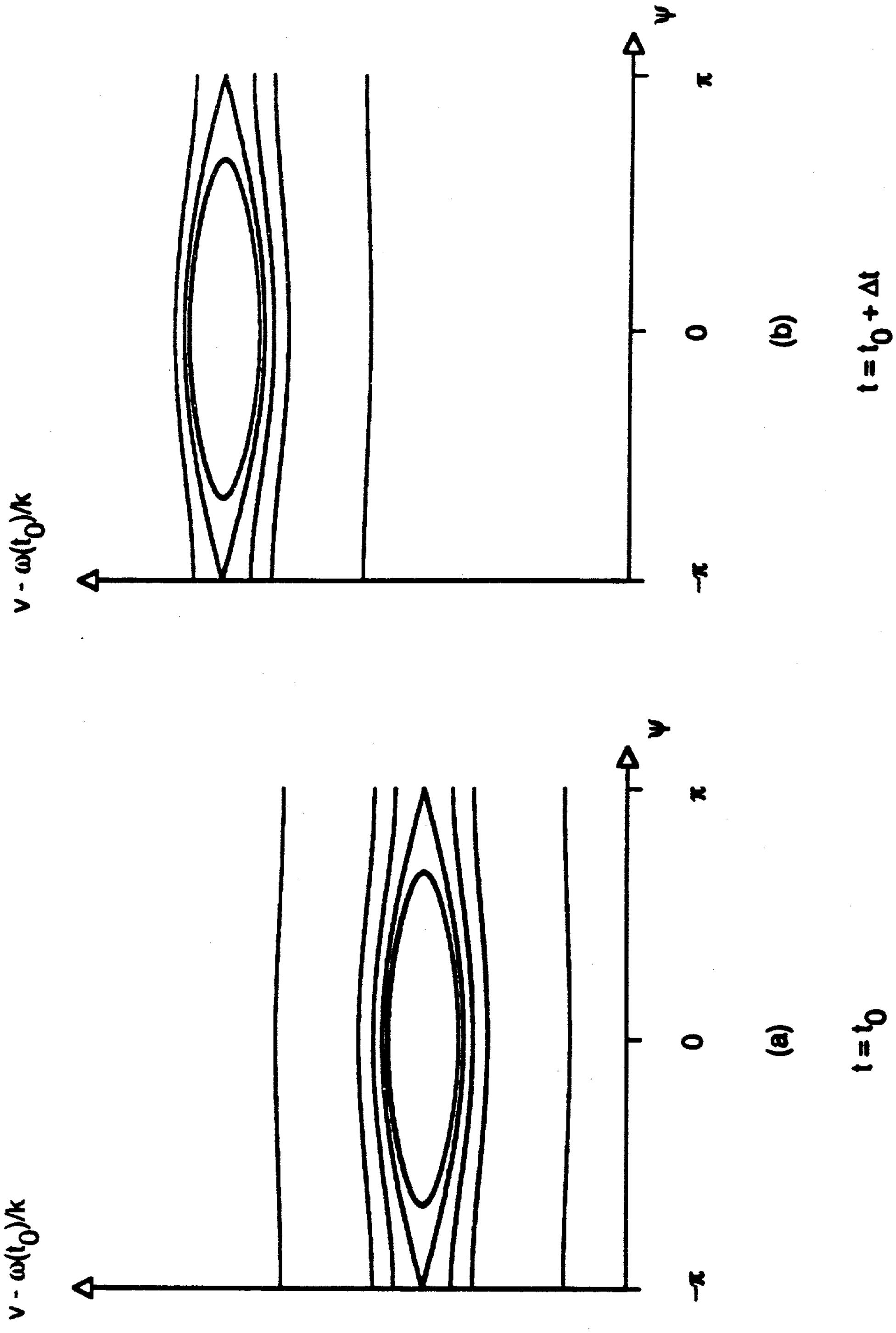


Fig. 1

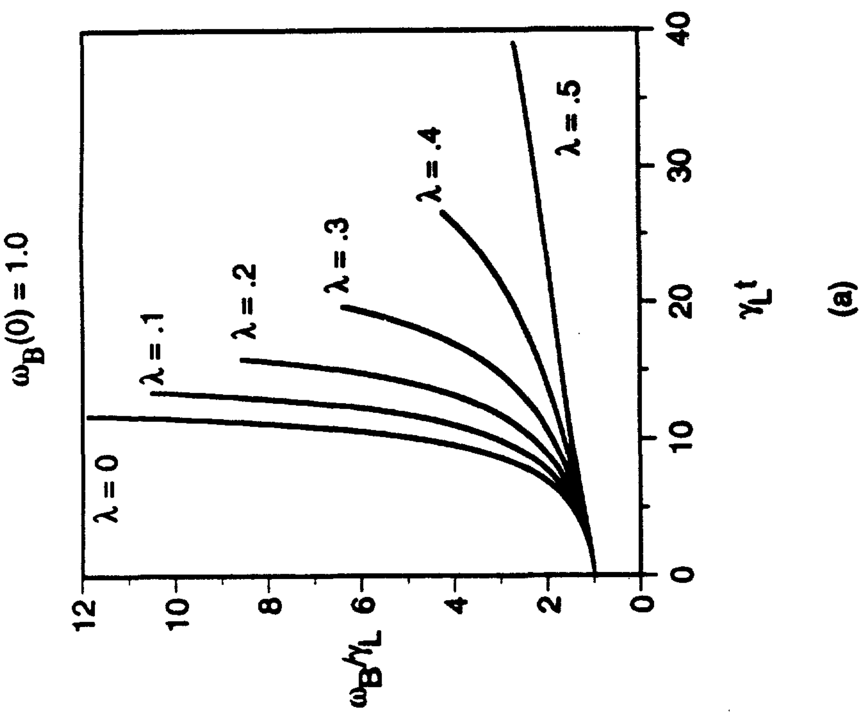


Fig. 2(a)

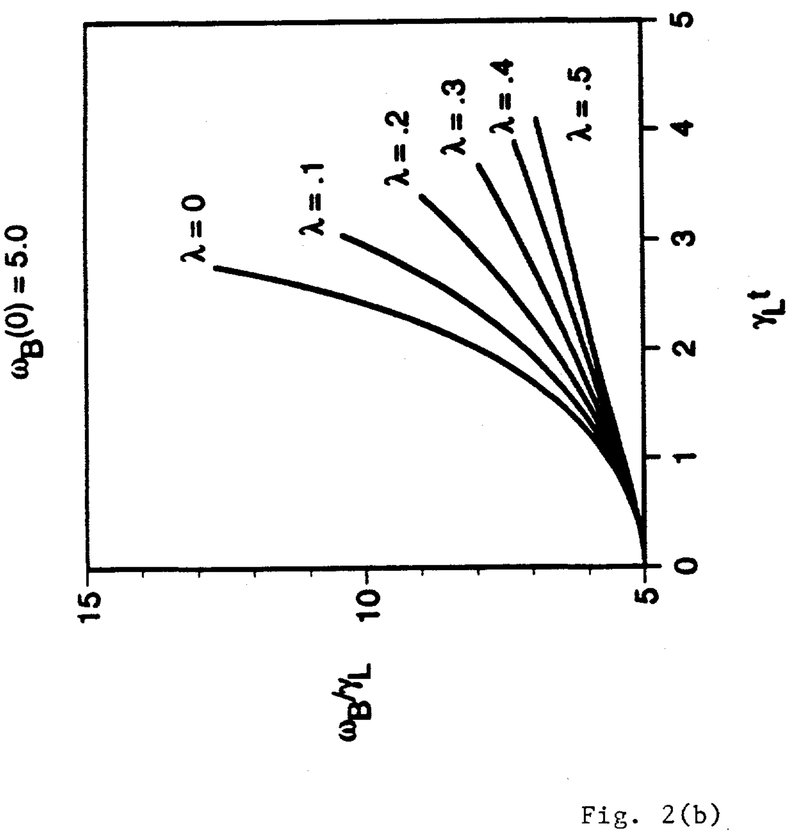


Fig. 2(b)

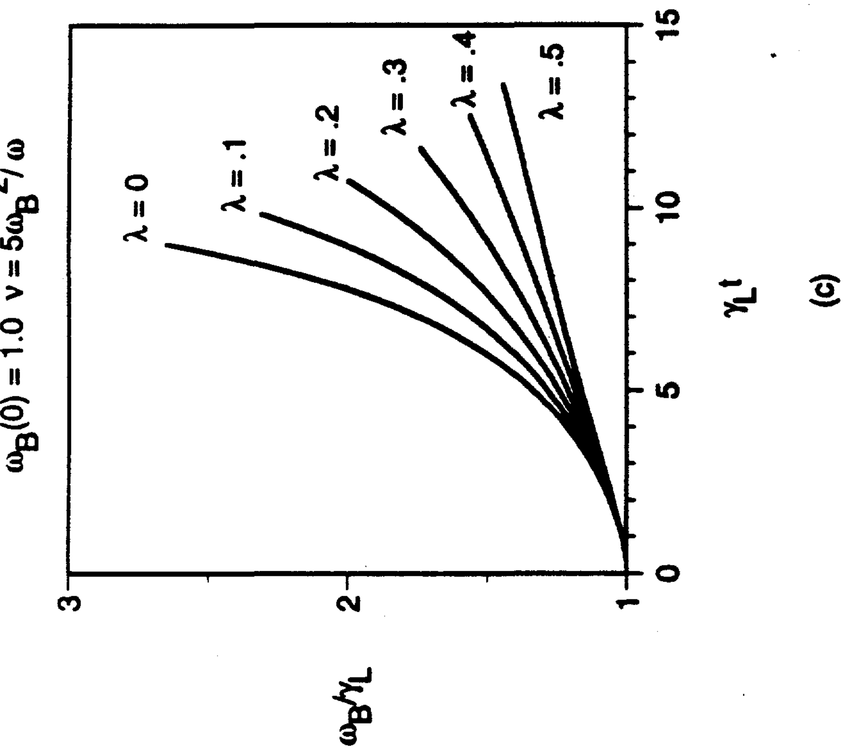


Fig. 2(c)