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TRANSPORT IN A STOCHASTIC MAGNETIC FIELD*

R.B. White, J.M. Rax**, Yanlin Wu

Plasma Physics Laboratory, Princeton University, Princeton N.J. 08543, USA

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Abstract

Collisional heat transport in a stochastic magnetic field configuration is investigated. Well above stochastic threshold, a numerical solution of a Chirikov-Taylor model shows a short-time nonlocal regime, but at large time the Rechester-Rosenbluth effective diffusion is confirmed. Near stochastic threshold, subdiffusive behavior is observed for short mean free paths. The nature of this subdiffusive behavior is understood in terms of the spectrum of islands in the stochastic sea.

1 Introduction: stochasticity, transport, and diffusion

A three dimensional toroidal magnetic field is a Hamiltonian system, and the existence of magnetic surfaces in an axisymmetric configuration is a consequence of the K.A.M stability theorem. The occurrence of small resonant magnetic field perturbations can lead to the onset of chaotic field line diffusion, provided that the Chirikov criterion is fullfilled [1].

This deconfinement mechanism is a major candidate to explain the anomalous confinement properties of Tokamak discharges [2,3].

This paper reviews some recent results, and reports some ongoing progress on the problem of heat transport in a stochastic magnetic field configuration.

Well above stochastic threshold, a numerical solution confirms the Rechester-Rosenbluth effective diffusion regime, but for short time a nonlocal regime is observed and studied analytically [4]. Near stochastic threshold, subdiffusive behavior is observed for short mean free paths. The nature of this subdiffusive behavior is understood in terms of the spectrum of islands in the stochastic sea.

The most general transport theory of a passive scalar, $T(\mathbf{r},t)$ (temperature or density), in a statistically homogeneous media, is nonlocal and relates T at point $\mathbf{r}$, time t, to the source, $S(\mathbf{r}',t')$, elsewhere before t, through a propagator P:

$$T(\mathbf{r},t) = \int dt'd\mathbf{r}'P(\mathbf{r}-\mathbf{r}',t-t')S(\mathbf{r}',t'). \tag{1}$$

P is the fundamental quantity arising through first principles, the diffusion coefficient is not fundamental and may or may not exist. Let us consider the long distance behavior of the Fourier transform, $P(\mathbf{k},\omega)$, of $P(\mathbf{r},t)$. If its asymptotic form for large $\mathbf{r}$ and t, i.e. near $\mathbf{k} \approx 0$, and $\omega \approx 0$, is $P^{-1}(\mathbf{k},\omega) \sim i\omega + \mathbf{k}.\mathbf{D}.\mathbf{k} + O[k^4]$, then the transport becomes local and diffusive on large space and time scales. T obeys the usual differential equation, and the mean-squared displacement $\langle \mathbf{r}(t)\mathbf{r}(t)\rangle = 2\,\mathbf{D}\,t$ is normal.

Recently a large number of physical problems such as diffusion in convective cells, diffusion on stochastic webs, diffusion on percollating clusters, convection of a passive scalar by a turbulent flow, and diffusion in disordered media, have displayed anomalous mean-squared displacement [5];

$$\langle r^2(t)\rangle \sim t^{\alpha}, \text{ and } \langle r^2(t)\rangle \sim \log^{\beta}[t]. \tag{2}$$

The exponent α, characterizing the fractional Brownian motion, can be larger or smaller than one, with trapping or long ballistic flights often the cause of this subdiffusive or hyperdiffusive transport. When $\alpha \neq 1$, an application of the definition of the diffusion coefficient will give 0 or ∞. This does not mean that the diffusion is 0 or $+\infty$, it means that the concept of diffusion coefficient is meaningless, and that the expansion of $P(\mathbf{k},\omega)$ does not give a simple formula.

On the other hand the occurrence of logarithmic behavior is often due to trapping. The case $\beta=4$ has been widely investigated in relation to Sinai's result of diffusion in a random one dimensional potential [6].

In this paper we report and analyze the observation of subdiffusive collisional transport in a stochastic magnetic field configuration, slightly above threshold.

2 Heat transport well above stochastic threshold

Because of its potential to explain anomalous confinement, heat transport in a stochastic magnetic field has been intensively investigated. Besides the thermal Larmor radii four length scales are relevant: the collisional mean free path along the magnetic field λ_C, the longitudinal and transverse autocorrelation lengths of the perturbing magnetic field, $\lambda_{||}$ and $\lambda_{\perp}$, and the Lyapunov length λ_K associated with the Kolmogorov entropy [7].

The existence of a finite λ_K allows description of the dynamics of a heat pulse in terms of combined collisional diffusion and exponential Lyapunov stretching, resulting in an effective diffusion $\langle r^2(t)\rangle = 2D_{RR}t$ first obtained by Rechester and Rosenbluth [2] and Stix [3], and recently numerically confirmed by Rax and White [4].

To set up a numerical simulation we consider the Chirikov-Taylor model. Starting from an unperturbed homogeneous magnetic field B directed along the z axis we add a shear $b_y = Bx$, and a multimode coherent pertubation $b_x = \varepsilon B \sin(y)[1+ 2\sum\cos(2\pi nz)]$, where the sum is over all positive integers n.

A Lagrangian representation of an Eulerian anisotropic heat flow along and across this magnetic configuration is achieved by adding a random reversal of the velocity v, along z correlated with a cross field displacement of gyroradius ρ. The Poincare section of such a dynamics for $v_t = 1$ is:

$$\begin{cases} z_{t+1} = z_t + v_t \\ y_{t+1} = y_t + x_{t+1} \\ x_{t+1} = x_t + \varepsilon \sin[y_t] + \rho_t \end{cases} \tag{3}$$

and for $v_t = -1$ Eqs. (2,3) must be inverted so that for $\rho_t=0$ a particle retraces its path back along the field line. The sign of v_t for each particle is changed each time step with probability P producing diffusive motion in z.

A collisional displacement ρ_t of magnitude ρ and random sign is given to x on those time steps that v_t changes sign. To achieve statistical accuracy typically 10^5 particles are used. The parallel mean free path is $\lambda_C = (1-P)/P$, and the parallel and perpendicular diffusions are $\chi_{||} = .5(1-P)/P$ and $\chi_{\perp} = .5\rho^2 P$. The field line diffusion χ_M, autocorrelation length $\lambda_{||}$, and the Lyapunov length λ_K are determined numerically. Because the Chirikov-Taylor model is defined on a torus the transverse autocorrelation length $\lambda_{\perp} = 2\pi$.

The results of our simulations can be summarized as follows. The long time behavior is found to be diffusive, $\langle r^2(t)\rangle = 2D_{eff}\, t$, with an effective diffusion coefficient D_{eff} in complete agreement with D_{RR}, but the short time behavior is found to be $\langle r^2(t)\rangle \approx \sqrt{t}$.

The usual ballistic regime near $t \sim 0$, $\langle r^2(t)\rangle \sim t^2$ is unobservable because of the scales. The transport in the diffusive regime is due to the finite Lyapunov length λ_K of the Chirikov-Taylor model: through the exponential stretching character of the stochastic instability deterministic chaos allows a rapid spread of a heat pulse. Following Rechester, Rosenbluth and Stix we have $D_{RR}=k\,\chi_M/\sqrt{\nu}$ where k is a constant and ν is the number of collisions necessary for decorrelation:

$$\lambda_K \ln[\lambda_\perp/\rho\sqrt{\nu}] = \lambda_C\sqrt{\nu}\,. \qquad (4)$$

We show here the region near stochastic threshold, of particular interest in plasma physics applications, but similar results hold in the large ε quasilinear domain provided accelerator modes are avoided. Excellent agreement is obtained with the single constant k=0.5 for very large ranges of P, ρ, and ε.

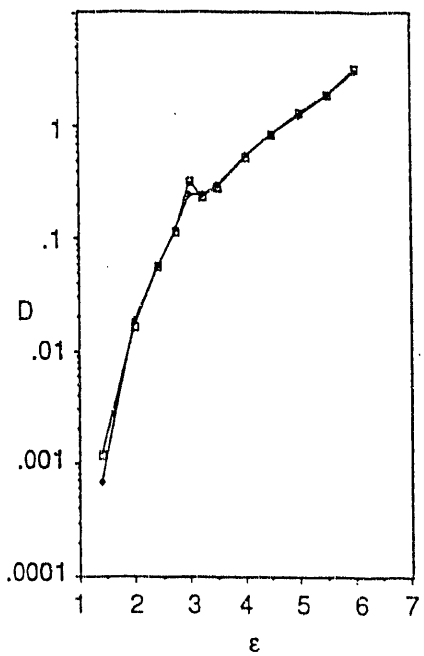

Fig. 1 The theoretical value $D_{RR}=k\,\chi_M/\sqrt{\nu}$, and the numerical one D_{eff} as a function of the stochasticity parameter ε. Here P = 0.1, $\rho = 10^{-8}$.

Besides this effective diffusion regime we have also observed subdiffusive behavior at short time. Such behavior can be easily analyzed with the help of a random Gaussian field model.

Consider an infinite homogeneous magnetic field **B** directed along the z axis, and a small random field, **b**, such that $\langle \mathbf{b}\rangle = 0$, $\nabla\cdot\mathbf{b} = 0$. $\langle b^2\rangle/B^2$ is a small parameter of the order of 10^{-6}-10^{-8} and $\langle\ \rangle$ indicates statistical averages.

The second moments of the **b** field are characterized by the longitudinal (z) correlation length, $\lambda_{||}$, $\langle b_x(z)b_x(z')\rangle = \langle b_y(z)b_y(z')\rangle = \langle b^2\rangle \exp[-(z-z')^2/2\lambda_{||}^2]$. The magnetic diffusion coefficient is $\chi_M = \sqrt{\pi/2}\,\lambda_{||}\langle b^2\rangle/B^2$.

The temperature diffusion equation is equivalent to Langevin equations with Lagrangian coordinates (x,y,z).

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{dz}{dt}\dfrac{b_x(z)}{B} + \eta_\perp(t) \\ \dfrac{dy}{dt} = \dfrac{dz}{dt}\dfrac{b_y(z)}{B} + \eta_\perp(t) \\ \dfrac{dz}{dt} = \eta_\parallel(t) \end{cases} \quad . \tag{5}$$

The statistical characteristics of the collisional noises η are $\langle\eta_\perp(t)\rangle=0$, $\langle\eta_\perp(t)\eta_\perp(t')\rangle=2\chi_\perp\delta(t-t')$, $\langle\eta_\parallel(t)\rangle=0$, $\langle\eta_\parallel(t)\eta_\parallel(t')\rangle=2\chi_\parallel\delta(t-t')$ (δ is the Dirac distribution). Thus when **b**=**0**, we recover collisional fast diffusion along **B**, and very slow diffusion across **B**. Let us introduce **b**(k), the Fourier transform of $\mathbf{b}(z)$, and integrate Eqs. (5) to obtain $\langle r^2(t)\rangle=\int\frac{dk}{k}\frac{dk'}{k'}\langle\frac{\mathbf{b}(k).\mathbf{b}^*(k')}{B^2}\rangle\langle[e^{ikz(t)}-1][e^{ikz(t)}-1]\rangle)+\int dudu'\langle\eta_\perp(u)\eta_\perp(u')\rangle$. Then using the Gaussian properties of the collisional noise $\langle e^{ikz(t)}\rangle=e^{-k^2\chi_\parallel t}$, and $\langle b_x(k)b_x^*)(k')\rangle = \langle b_y(k)b_y^*(k')\rangle = \langle b^2\rangle\lambda_\parallel\delta(k-k')\exp[-k^2\lambda_\parallel^2/2]/\sqrt{2\pi}$ we obtain $\langle r^2(t)\rangle = 8\chi_M\sqrt{\chi_\parallel t/\pi} + 4\chi_\perp t$.

No effective diffusion coefficient is found with this exact solution of the Langevin equation and the subdiffusive behavior obtained within the framework of this model agrees qualitatively and quantitatively with the short-time behavior observed with the map model. In the nonlocal regime the propagator to be used in Eq. (1) can be calculated with the help of a Wiener functional integration and the final result is:

$$P(r,t) = \Theta(t)\exp\left[-\frac{3r^{4/3}}{2^{2/3}4\chi_M^{2/3}\chi_\parallel^{1/3}t^{1/3}}\right]/2\pi 2^{2/3}r^{2/3}\chi_M^{2/3}\chi_\parallel^{1/3}t^{1/3}. \tag{6}$$

The anomalous scaling of the exponent agrees with the previous mean square displacement, and this propagator basically describes a random walk along a random field.

3 Heat transport near stochastic threshold

Of particular interest for magnetic confinement is the behavior near stochastic threshold, the most probable state of magnetic fields in tokamak devices. Near threshold, for very short mean free path, the previous Chirikov-Taylor model displays the occurence of subdiffusive behavior,

$$\langle x^2(t)\rangle \approx \log^p[t] \quad . \tag{7}$$

In the following we will describe how the spectrum of islands in the stochastic sea is responsible for this result.

A random walk can always be decomposed as a set *step/pause/step/pause/step/pause*, and so on. To calculate the mean square displacement as a function of time we have to know two basic quantities, namely the distribution function of step size $\phi(s)$ and the the waiting time distribution $\psi(t)$ at the pause. Then given an N step walk we can write:

$$\begin{cases} <x^2> = <[\sum_1^N s_i]^2> = N \int_0^{s_o} s^2\, \phi(s)\, ds \\ <t> = <\sum_1^N t_i> = N \int_0^{t_o} t\, \psi(t)\, dt \end{cases} \qquad (8)$$

To express the mean square displacement as a function of the time we have to eliminate N from these equations. In a given experiment the particles may not have time to completely explore the distribution of step sizes, even if this distribution is bounded . Thus rather than taking $s_o = t_o = +\infty$, we have to introduce two cut off points which are in fact N dependent.

This regularization, depending on the behavior of ϕ and ψ at large argument, can be responsible for the occurrence of subdiffusive behavior, such as given by Eq. (2).

To examine the problem of transport near threshold in the Chirikov map model, the evaluation of these two functions requires a detailed study of the distibution of island sizes and of the trapping near islands. Roughly speaking, wandering closely around an island is the pause and the island size the step.

To calculate the distribution function between the scale $s \approx 0$ and $s \approx 1$ we proceed as follows. Introduce a grid, and calculate the extent of the stochastic sea by initially setting the function $n = 1$ everywhere and then following a single long orbit in the stochastic sea, setting $n = 0$ for each grid square visited. Using this technique it is possible to distinguish islands from stochastic sea for all islands larger than the grid size, and the method has been used by Umberger and Farmer to investigate the distribution of island sizes.

For small islands the distribution of the islands is fractal, being given by a power law, with the area of all islands of size greater than s being given by

$$A(s) = A(0) - c\, s^p \; . \qquad (9)$$

We are interested in accurate determination of areas also for large islands, so the covering methods described by them are not sufficient. However an accurate determination of the distribution of island areas and perimeters can be made from the function n(i,j).

Scanning the grid, each connected set of island points can be labeled by setting $n(i,j) = k$, with k an index labeling that island, and the next connected set of points encountered labeled by $k+1$. Finally the islands at the edges of the domain are joined in accordance with topological identification on the torus, to insure that the results are not dependent on the location of the edges of the domain.

Then by counting the number of points with index k to determine the area of island k, a complete catalogue of islands can be made, each with its associated area and perimeter, from the island of maximum size s_0 down to islands of the grid size. For this purpose we define the size to be the square root of the island area. Sample results are shown in Fig. 2.

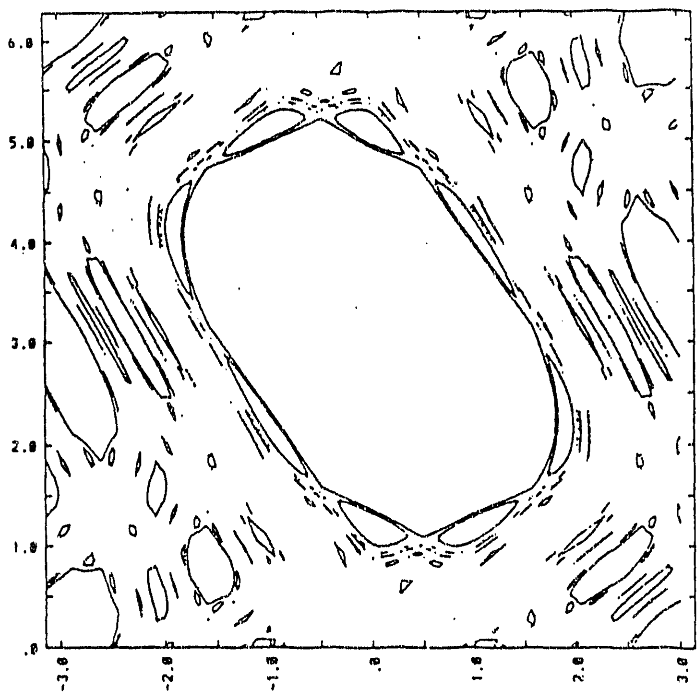

Fig. 2 island perimeters found using a grid mesh of 1000x1000, $\varepsilon = 1.1$

Using continuum notation, the expression for the total area of all islands with size greater than s is

$$A(s) = \int_{s}^{s_0} s^2 \phi(s)\, ds, \tag{10}$$

with $\phi(s)$ the density of islands. The total area occupied by the islands, $A(0)$ is finite, but the total number of islands is infinite. Results are shown in Fig. 3.

The results for small s agree with those obtained by Umberger and Farmer [8], Eq. (9), with $p \approx 0.6$. For $s \approx s_0$ the scaling is quite different, and we observe $A(s) \approx s^{-a}$ with $a = 0.2$.

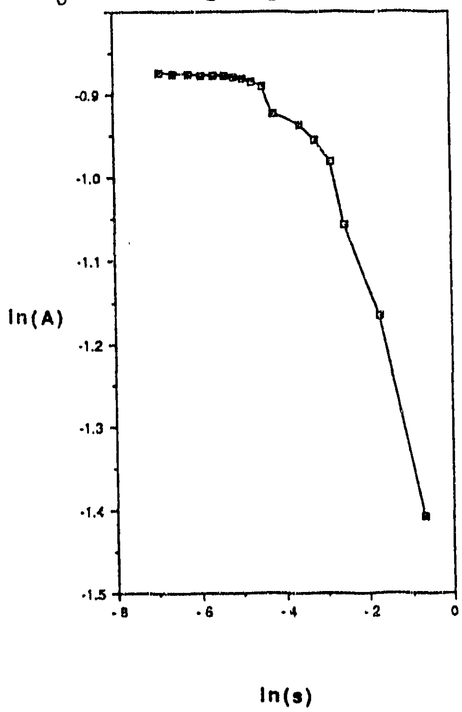

Fig.3 the distribution of island size, note the occurence of two scaling regimes.

From Eq. (10) we find, in this domain, $\phi(s) \approx s^{-3.2}$. We refer to this scaling regime as the dynamical range, as we will see that it determines the short time collisional diffusion of the medium. The transition between the two regimes occurs approximately at island sizes corresponding to the smallest islands visible in Fig. 2.

Now we proceed to evaluate $\psi(t)$. By observing single particle orbits it is found that long periods of time are spent circling the periphery of the largest islands. In fact near threshold, when the islands fill a significant part of the stochastic sea, the dominant contribution to the time spent in any orbit comes from circling the largest island.

By measuring this time, preliminary results indicate that the island trapping time τ depends exponentially on the island perimeter, i.e., $\tau(s) = \tau_0 \, e^{s/s_0}$.

Diffusion then consists of a step with size given by the linear dimension of an island, and subsequent wandering in the stochastic sea until the particle is again attached to an island. Near stochastic threshold, with many islands present, this latter phase occupies a negligible amount of time. The integration limits in Eq. (8) depend on the amount of time in the simulation, and therefore how much of the island distribution the particles have been able to explore.

Since there is in fact a largest island of scale s_0, if one waits long enough the particles will explore all the islands, the integrals are independent of N, the ratio of $<x^2>$ to $<t>$ is independent of N, and the result is diffusive motion.

This result requires lengths of time in which the total distance diffused is greater than the basic periodicity of the system, 2π. However, for shorter times the limits of these integrals s_0, t_0 must be replaced by the largest values of s, t which the system has had time to explore, s_m and t_m. Since the truncation of the integrals depends on the number of steps N, the values of s_m and t_m depend on N and the motion is not necessarily diffusive. To find s_m and t_m consider N random selections of islands using the distribution ϕ. The limit s_m, and hence the large scale islands not visited will be given by the expression

$$\int_{s_m(N)}^{s_o} \phi(s)\, ds = \frac{1}{N} \text{ ,i.e., } s_m = \left[\frac{N}{a+2}\right]^{1/a+2} \quad . \tag{11}$$

The trapping time depends exponentially on the island size, we have $t_m = e^{s_m(N)}$ The distribution of trapping times is also related to the distribution of the islands through

$$\Psi(t) = \int \delta[t - \tau(s)]\, \phi(s)\, ds, \quad \text{i.e.,} \quad \psi(t) = \frac{s_o^{a+2}}{t \log^{a+3}(t/\tau_o)} \tag{12}$$

where we have used $\tau(s) = \tau_o e^{s/s_o}$. Substituting and evaluating Eqs (6) and (7), after N steps the dominant scaling is

$$<x^2> \approx N, \quad <t> \approx e^{N^{1/a+2}}, \text{ i.e.,} \quad <x^2> \approx \log^{a+2}(t) \quad . \tag{13}$$

Using the measured value a=0.2 this result agrees quite well with the observed subdiffusive behavior. Thus the decomposition analysis of the behavior of transport in term of island trapping appears to be consistent with the observed logaritmic mean squared displacement. The structural reduction *step/pause/step/pause/step....* is thus valid provided we take the two elementary processes to be *pause* = (wandering around an island), and *step* = (jumping between islands).

4 Conclusion

We have reviewed some recent results, and reported some ongoing progress on the problem of heat transport in a stochastic magnetic field configuration.

The collisional diffusion coefficients scale differently with respect to density n, and temperature T: $\chi_{\parallel} \sim T^{5/2} n^{-1}$, and $\chi_{\perp} \sim T^{-1/2} n$, so that in tokamaks we obtain a wide range of values: $\chi_{\parallel} \sim 10^9$-$10^{10} m^2/s$, and $\chi_{\perp} \sim 10^{-1}$-$10^{-2} m^2/s$. The typical width of braided magnetic domain is $r \sim 10^{-2}$-1.m, and $\chi_M \sim 10^{-5}$-10^{-8}m. This wide range of values leads to the potential occurence of a wide range of regimes, some of which have been investigated here.

We find using the Chirikov map model that Rechester Rosenbluth diffusion is valid, and quite accurate over a very large domain in parameter space. Complete results will be presented in a future publication. We are presently extending these investigations to include other stochastic configurations. Also of great interest is the extension of this work to include two species incorporating momentum conserving collision operators.

Subdiffusive behaviour is observed near stochastic threshold. The results presented here are preliminary, however the identification of the waiting time arround an island and the self-similar spectrum of island size appear to be the two basic ingredients required to understand transport near the stochastic threshold.

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